M2 THESIS: AROUND P. DELIGNE'S COMPANION CONJECTURE

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Abstract

Motivated by the idea that local systems should only exist for some "motivic reason", P. Deligne formulated the companion conjecture [Del80, Conjecture 1.2.10] for normal schemes. Based on L. Lafforgue's proof for curve, we will introduce various works around this conjecture (by P. Deligne, V. Drinfeld, H. Esnault, M. Kerz and A. Cadoret), in particular a proof of the companion conjecture for smooth schemes using "skeleton sheaves".

Key words: P. Deligne's companion conjecture, ℓ-independence, ℓ-adic local system, ℓ-adic representation, skeleton sheaves, weakly motivic.

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1 Introduction

1.1 ℓ-Independence and Motives

(1.1.1) Let $\mathbb{F}_q$ be a finite field with characteristic p, $\mathbb{F}$ a separable closure of $\mathbb{F}_q$, ℓ a prime number $\neq p$. Let X_0 be a scheme separated and of finite type over $\mathbb{F}_q$ and $X = X_0 \times_{\mathbb{F}_q} \mathbb{F}$. Consider cohomology groups with compact supports $H^i_c(X, \mathbb{Q}_\ell)$. They are finite dimensional $\mathbb{Q}_\ell$-vector spaces, zero for $i > 2\dim X_0$, acted by Galois group $G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. Consider the ℓ-adic numbers

$$\mathrm{Tr}\left(g, H^*_c(X, \mathbb{Q}_\ell)\right) = \sum (-1)^i \,\mathrm{Tr}\left(g, H^i_c(X, \mathbb{Q}_\ell)\right) \tag{1.1}$$

for $g \in G$. A natural question is : is this sum independent of the choice of ℓ?

(1.1.2) Take $g = 1$, then the sum (1.1) is the Euler-Poincaré characteristic (with compact supports)

$$\chi_c(X, \mathbb{Q}_\ell) = \sum (-1)^i \dim H^i_c(X, \mathbb{Q}_\ell).$$

Let F be the geometric Frobenius, by Grothendieck's trace formula (A.1.10)

$$Z(X_0, T) = \prod \det\left(1 - FT, H^i_c(X, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}},$$

we have $\chi_c(X, \mathbb{Q}_\ell) = -\deg Z(X_0, T)$, thus follows the independence of ℓ.

(1.1.3) Consider the Betti numbers

$$b^i(X, \mathbb{Q}_\ell) = \dim H^i(X, \mathbb{Q}_\ell),$$
$$b^i_c(X, \mathbb{Q}_\ell) = \dim H^i_c(X, \mathbb{Q}_\ell).$$

If X_0 is proper and smooth, we can use Weil conjectures and make induction on dimension of X_0 by choosing a Lefschetz pencil to get that, for each i, $b^i(X, \mathbb{Q}_\ell)$ is independent of ℓ, and actually coincides for any Weil cohomology [KM74, Corollary 1].

(1.1.4) We consider

$$P^i_\ell(T) := \det\left(1 - TF \mid H^i(X, \mathbb{Q}_\ell)\right),$$
$$P^i_{c,\ell}(T) := \det\left(1 - TF \mid H^i_c(X, \mathbb{Q}_\ell)\right).$$

Since $\deg\left(P^i_\ell\right) = b^i(X, \mathbb{Q}_\ell), \deg\left(P^i_{c,\ell}\right) = b^i_c(X, \mathbb{Q}_\ell)$, the independence of ℓ of $b^i_c(X, \mathbb{Q}_\ell)$ follow from the independence of $P^i_\ell(T)$ and $P^i_{c,\ell}(T)$. In general, the independence of ℓ of $P^i_\ell(T)$ and $P^i_{c,\ell}(T)$ is unknown. But P. Mannisto and M. Olsson [MO] show the independence for dimension of $X_0 \leq 2$.

(1.1.5) Suppose that X_0 and Y_0 are both proper and smooth and that we are given a $\mathbb{F}_q$-morphism $f\colon Y_0 \to X_0$. For each integer $i \geq 0$ and each $\ell \neq p$, we have an induced map (because f is proper) which is Galois-equivariant

$$(f^*)_{i,\ell} : H^i_c(X, \mathbb{Q}_\ell) \to H^i_c(Y, \mathbb{Q}_\ell).$$

It is conjectured that the characteristic polynomials of F on both the kernel and cokernel of $(f^*)_{i,\ell}$ have $\mathbb{Z}$-coefficients, independent of $\ell \neq p$.

Now let's introduce the formalism of "Motives".

(1.1.6) Let $\mathbb{F}_q$ be a finite field with characteristic p. Let $\mathrm{Sch}/\mathbb{F}_q$ be the category of normal schemes separated and of finite type over k. Grothendieck expected that attached to any $X_0 \in \mathrm{Sch}/\mathbb{F}_q$, we have a graded semisimple $\mathbb{Q}$-linear rigid abelian $\otimes$-category $M(X_0, \mathbb{Q})$, with $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$ if X_0 is connected.

Grothendieck expected several properties satisfied by $M(X_0, \mathbb{Q})$ [EK11, §2]. Let's clarify two properties here:

1. For any prime number ℓ different from p, there is a faithful $\mathbb{Q}_\ell$-linear $\otimes$-functor

$$\mathrm{R}_\ell : M(-, \mathbb{Q}) \otimes \mathbb{Q}_\ell \to \mathrm{Sh}(-, \mathbb{Q}_\ell),$$

   where $X \mapsto \mathrm{Sh}(X_0, \mathbb{Q}_\ell)$ is the étale stack of lisse $\mathbb{Q}_\ell$-étale sheaves over $\mathrm{Sch}/\mathbb{F}_q$.

2. There is a contravariant functor from the category of smooth projective schemes $f : Y_0 \to X_0$ to motives $\mathfrak{h}(Y_0) \in M(X_0, \mathbb{Q})$ such that

$$\mathrm{R}_\ell \circ \mathfrak{h}(Y) \cong \bigoplus_n R^n f_* \mathbb{Q}_\ell.$$

(1.1.7) For any field $F \supset \mathbb{Q}$, let us define $M(X_0, F)$ to be the pseudo-abelian envelope [Stacks, Tag 09SF] of $M(X_0, \mathbb{Q}) \otimes F$.

(1.1.8) Suppose that Grothendieck standard conjecture D holds, i.e. numerical equivalence coincides with $\mathbb{Q}_\ell$-cohomological equivalence for every $\ell \neq p$. Then motives for numerical equivalence have ℓ-adic realizations for every $\ell \neq p$. According to U. Jannsen [Jan92], the category of motives for numerical equivalence is abelian and semisimple (actually, the proof only uses Wedderburn's theorem and linear algebra). For a given variety proper and smooth over a finite field, the Künneth components of the diagonal are rationally algebraic, represented by universal (i.e., independent of $\ell \neq p$) $\mathbb{Q}$-linear combinations of the graphs of iterates of Frobenius [KM74, Theorem 2, 1)]. So the individual cohomology groups $H^i_c(X, \mathbb{Q}_\ell)$ and $H^i_c(Y, \mathbb{Q}_\ell)$ are the ℓ-adic realizations of motives. By U. Jannsen [Jan92], the corresponding motivic kernels and cokernels of f^* exist. Our ℓ-adic kernels and cokernels are the ℓ-adic realizations of these motivic kernels and cokernels. Then (1.1.4) follows by [KM74, Theorem 2, 2)].

Conjetcture (1.1.9) The essential image of

$$\mathrm{R}_\ell : M\left(X_0, \overline{\mathbb{Q}}_\ell\right) \to \mathrm{Sh}\left(X_0, \overline{\mathbb{Q}}_\ell\right)$$

consists of direct sums of irreducible sheaves V which are pure of integral weight, and such that the eigenvalues of geometric Frobenius F_x for all closed points $x \in |X_0|$ are ℓ'-adic units for all prime numbers $\ell' \neq p$.

The motivic expectation above motivates P. Deligne's companion conjecture [Del80, 1.2.10].

1.2 L. Lafforgue's Langlands Theorem

(1.2.1) We follow the notation in [Cad18]. If Q is an algebraic extension of $\mathbb{Q}_\ell$, by a Q-coefficient we mean a lisse Weil Q-sheaf. Similarly, by a $\overline{\mathbb{Q}}_\ell$-coefficient we mean a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf. A Weil $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{C}_0$ is said to be algebraic if for any x in any $X(\mathbb{F}_{q^n})$, $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ is algebraic number.

(1.2.2) Using the global Langlands correspondence (between cuspidal automorphisms and irreducible $\overline{\mathbb{Q}}_\ell$-coefficients with finite determinant), L. Lafforgue proved a theorem for smooth curves [Laf02, Theorem VII.6]:

Theorem (1.2.3) Let $X_0/\mathbb{F}_q$ be a smooth curve, let V_0 be a $\overline{\mathbb{Q}}_\ell$-coefficient on X_0 with determinant of finite order. Then:

(i) There exists a field Q_{V_0} which is a finite extension of $\mathbb{Q}$ and for all $x \in |X_0|$, we have $\chi_x(V_0, T) := \det\left(Id - TF_x \mid V_x\right) \in Q_{V_0}[T]$, where F is the geometric Frobenius.iii90o

(ii) For an arbitrary, not necessarily continuous, automorphism $\sigma \in \mathrm{Aut}\left(\overline{\mathbb{Q}}_\ell/\mathbb{Q}\right)$, there is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-Weil sheaf V_0^σ on X, called σ-companion, with determinant of finite order such that

$$\chi_x(V_0^\sigma, T) = \sigma\left(\chi_x(V_0, T)\right),$$

where σ acts on the polynomial ring $\overline{\mathbb{Q}}_\ell[T]$ by σ on $\overline{\mathbb{Q}}_\ell$ and by $\sigma(T) = T$.

(iii) V_0 is pure of weight 0 .

This was already expected by P. Deligne in [Del80] since at that time the Langlands correspondence was available for $n = 2$, $\ell \neq p$ by the work of V. Drinfeld.

(1.2.4) On higher dimensional schemes, unfortunately, there seems to be no analogue of the Langlands correspondence (even conjectural) which would provide geometric origins for lisse sheaves on higher-dimensional varieties, except in the case of weight 1 where some results are known.

1.3 P. Deligne's Companion Conjecture

(1.3.1) Motivated by the idea that local systems should only exist for some "geometric reason", P. Deligne conjectured [Del80, Conjecture 1.2.10] that every $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$ on a separated normal scheme of finite type over $\mathbb{F}_q$ admits a $\overline{\mathbb{Q}}_{\ell'}$-companion $\mathcal{C}_0'$.

Definition (1.3.2) Let $\mathcal{C}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient and let $\mathcal{C}_0'$ be a $\overline{\mathbb{Q}}_{\ell'}$-coefficient. Fixing two isomorphisms $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ and $\iota' : \overline{\mathbb{Q}}_{\ell'} \xrightarrow{\sim} \mathbb{C}$, we say that $\mathcal{C}_0'$ is a $\overline{\mathbb{Q}}_{\ell'}$-companion of $\mathcal{C}_0$ if for any $x \in |X_0|$, $\chi_x(\mathcal{C}, T)$ coincides with $\chi_x(\mathcal{C}', T)$, i.e.

$$\iota' \det\left(Id - TF_x \mid \mathcal{C}_x'\right) = \iota \det\left(Id - TF_x \mid \mathcal{C}_x\right).$$

Conjetcture (1.3.3) Let X_0 be a normal scheme of finite type over $\mathbb{F}_q$, $\ell \neq p$ a prime, and $\mathcal{C}_0$ an irreducible $\overline{\mathbb{Q}}_\ell$-coefficient on X_0 whose determinant has finite order.

(a) For some number field $Q_{\mathcal{C}_0}$, $\mathcal{C}_0$ is $Q_{\mathcal{C}_0}$-algebraic: for all x, we have $\chi_x(\mathcal{C}, T) \in Q_{\mathcal{C}_0}[T]$. The $Q_{\mathcal{C}_0}$ will be called trace field of $\mathcal{C}_0$.

(b) $\mathcal{C}_0$ is pure of weight 0: for every algebraic embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, for all x, the roots of $\chi_x(\mathcal{C}, T)$ in $\mathbb{C}$ all have complex absolute value 1 (one can avoid having to embed $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$ by (a)).

(c) For all x, the roots of $\chi_x(\mathcal{C}, T)$ have trivial λ-adic valuation at all finite places of λ of Q not lying above p.

(d) For every place λ of Q above p, for all x, the roots of $\chi_x(\mathcal{C}, T)$ have λ-adic valuation at most $\frac{1}{2}\operatorname{rank}(\mathcal{C})$ times the valuation of $\#\kappa(x)$ (the order of the residue field).

(e) For every prime $\ell' \neq p$, there exists a $\overline{\mathbb{Q}}_{\ell'}$-coefficient $\mathcal{C}'_0$ which is irreducible with determinant of finite order and which is a companion of $\mathcal{C}_0$.

Remark (1.3.4)

1. If a given coefficient $\mathcal{C}_0$ is pure, its trace field $Q_{\mathcal{C}_0} \subset \overline{\mathbb{Q}}$.
2. By Proposition (A.1.8), after some twist we can always assume the determinant of the lisse sheaf has finite order.
3. The uniqueness of ℓ'-adic companions up to semi-simplification is unique by applying the Čebotarev density theorem to mod-ℓ^n representations as in the proof of [Laf02, Proposition VI.11].
4. By [Laf02], part (b) of (1.3.3) holds iff the following holds: for any $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$ pure of weight 0, there exists an number field Q such that for all x, we have $\chi_x(\mathcal{C}, T) \in Q[T]$. Actually, if part (b) of (1.3.3) holds, then [Laf02] proves that $\mathcal{C}_0$ is pure of weight 0. Conversely, replace $\mathcal{C}_0$ by its semi-simplification and assume its irreducibility, there there exists a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{W}_0$ of rank 1 and of weight 0 on $\operatorname{Spec}\mathbb{F}_q$ such that $\det(\mathcal{C}_0 \otimes \mathcal{W}_0)$ is of finite order. Then the statement follows from the fact that $\operatorname{Tr}(F_x, \mathcal{C}_0 \otimes \mathcal{W}_0) = \operatorname{Tr}(F_x, \mathcal{C}_0) \cdot \operatorname{Tr}(F_x, \mathcal{W}_0)$.

1.4 P. Deligne's Proof of Finiteness

In Chapter (2), we will give a proof of part (a) of conjecture (1.3.3) due to P. Deligne himself [Del12]. We provide a sketch of proof here.

(1.4.1) Firstly for curve case, we consider an absolutely irreducible $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$ on a smooth curve X_0 defined over $\mathbb{F}_q$. WLOG, we can assume that X_0 is affine. The "complexity" of X_0 will be measured by the integer $b_1(X) := \dim H^1_c(X, \mathbb{Q}_\ell)$. We assume that $\mathcal{C}_0$ is algebraic (1.2.1). Our goal is to determine an integer N, depending on the "complexity" of $(X, \mathcal{C})$, such that E is generated by the $\operatorname{Tr}(F_x, \mathcal{C}_0)$ for x in $X_0(\mathbb{F}_{q^n})$ with $n \leq N$ and it contains all traces of $\mathcal{C}_0$. Let $\log_q^+(a) = \max(0, \log_q(a))$. If $\mathcal{C}$ is tamely ramified, we obtain N of the form

$$O\left(\log_q^+\left(b_1(X)\right)\right) + O(1)$$

where the implicit constants in O depend on the rank of $\mathcal{C}_0$. For a general $\mathcal{C}_0$, we replace $b_1(X)$ with $b_1(X) + \sum_{s\in S} \alpha_s(\mathcal{C})$, where $\alpha_s(\mathcal{C})$ is the slope (B.2.3) which measures the wildness of the ramification of $\mathcal{C}$ at s.

(1.4.2) For the general case, P. Deligne uses several dévissages and use N. Katz's estimation of Betti number [Kat01]. But in section (2.3) we will also give an alternative proof by H. Esnault and M. Kerz [EK12], which uses ramification theory to provide a suitable bound N and then applies a Deligne-Fourier transformation to get the result.

1.5 V. Drinfeld's Theorem

Using L. Larfforgue's result for curve, V. Drinfeld proves the companion conjecture for smooth case.

Theorem (1.5.1) Let X_0 be a smooth scheme over $\mathbb{F}_q$. Let Q be a finite extension of $\mathbb{Q}$. Let λ, λ' be nonarchimedean places of Q prime to p and let Q_λ, $Q_{\lambda'}$ be the corresponding completions. Let $\mathcal{C}_0$ be a $\overline{Q}_{\lambda'}$-coefficient on X_0 such that for every closed point $x \in X_0$ the polynomial $\chi_x(\mathcal{C}, T)$ has coefficients in Q and its roots are λ-adic units. Then there exists a $\overline{Q}_\lambda$-coefficient on X_0 compatible with $\mathcal{C}_0$.

Remark (1.5.2)

1. By P. Deligne's conjecture [Del80, Conjecture 1.2.10], the above theorem should hold for normal scheme.
2. By the definition, the $\overline{Q}_\lambda$-coefficient is defined over some finite extension L/Q_λ, and from the proof we will know that L is only determined by the rank of $\mathcal{C}_0$ and Q_λ [Dri18, Lemma 2.7].

(1.5.3) The key object we will define in Chapter (3) is 2-skeleton sheaf. This concept is used by V. Drinfeld [Dri18] and implicitly present in [Del12], but the terminology is introduced in [EK12] and credited to L. Kindler. Roughly speaking, 2-skeleton sheaf of a given smooth scheme X_0 is the data of coefficients arising from all curves in X_0. The idea is to compare the skeleton sheaf with local system arising from X_0, and in smooth case, we can find a subset of skeleton sheaf on X_0 called geometric skeleton (3.2) which is bijective to the coefficients arising from X_0.

The proof proceeds in three steps: the first step (4.1.2) is using compactness argument given by M. Kerz, the second step (4.1.3) will use the results of tame fundamental group from [SGA1] , and the third step (4.1.4) is more geometric and uses Bertini theorem.

1.6 Summary

Given two isomorphisms $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ and $\iota' : \overline{\mathbb{Q}}_{\ell'} \xrightarrow{\sim} \mathbb{C}$, combining all the results above, now we get the companion theorem:

Theorem (1.6.1) Let X_0 be a smooth variety, separated and of finite type over $\mathbb{F}_q$. Let $\mathcal{C}_0$ be an irreducible $\overline{\mathbb{Q}}_\ell$-coefficient with finite determinant. Then:

1. $\mathcal{C}_0$ is pure of weight 0;
2. $Q_{\mathcal{C}_0}$ is a finite extension of $\mathbb{Q}$;
3. There exists an étale $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0'$ which is compatible (with respect to ι, ι') with $\mathcal{C}_0$.

We will give two applications of the companion theorem in chapter (6), (7).

(1.6.2) The Čebotarev density theorem plays a fundamental part in arithmetic geometry in that it often enables to reduce problems about $\overline{\mathbb{Q}}_\ell$-local systems on X to problems about semisimple $\overline{\mathbb{Q}}_\ell$-local systems on points. We will prove Tannakian Čebotarev density theorem for semisimple coefficient in chapter (6). For étale coefficient, the theorem just follows from classical Čebotarev density theorem, and A. Cadoret reformulates Theorem (6.0.2) in terms of the characteristic polynomial map attached to $\mathcal{C}_0$ [Cad18, proposition 4.4] and proves the theorem for semisimple coefficients using companion theorem.

(1.6.3) The second application is to the theory of weakly motivic sheaves introduced by V. Drinfeld [Dri18]. P. Deligne defined $\mathscr{D}_{\text{mix}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ as the category of mixed $\overline{\mathbb{Q}}_\ell$-complexes in [Del80, §6.2.2], and by [BBD82] and [Del80],

$\mathscr{D}_{\mathrm{mix}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ is stable under "six operators". Using the companion theorem (1.6.1), we define the category of weakly motivic sheaves $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ in (7) and prove it's also stable under "six operators".

2 P. Deligne's Proof of Finiteness

In [Del12], P. Deligne proved the Conjecture (1.3.3)(b):

Theorem (2.0.1) Let X_0 be a scheme of finite type over $\mathbb{F}_q$. If $\mathcal{C}_0$ is an algebraic Weil $\overline{\mathbb{Q}}_\ell$-sheaf on X_0, then there exists a finite extension $Q \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}$ such that for every n and every $x \in X_0\left(\mathbb{F}_{q^n}\right)$, $\chi_x(\mathcal{C}, T) \in Q[T]$.

Remark (2.0.2) Using the identity between formal power series:

$$\log\det(1 - ft, V) = -\sum_{n\geq 1} \mathrm{Tr}\left(f^n, V\right)\frac{t^n}{n},$$

where V is a linear representation and f is an endomorphism of V. For the theorem above, it is equivalent to show that for every n and every $x \in X_0\left(\mathbb{F}_{q^n}\right)$, $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ is in Q.

2.1 Curve Case

(2.1.1) Let X_0 be a smooth, affine curve over a finite field, and let X be the scalar extension to algebraic closure, let $\mathcal{C}_0$ be an algebraic $\overline{\mathbb{Q}}_\ell$-coefficient of rank r. Denote:

$$N_0 = 2\log_q^+\left(2r^2\left(b_1(X) + \sum \alpha_x(\mathcal{C})\right)\right),$$
$$N = \lfloor N_0 \rfloor + 2r,$$

where the function $\log_q^+$ denotes $\sup\left(0, \log_q\right)$.

In this section, we prove:

Proposition (2.1.2) Let Q be the extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ generated by the $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ for x in $X_0\left(\mathbb{F}_{q^n}\right)$ with $n \leq N$. Then, for any n and any $x \in X_0\left(\mathbb{F}_{q^n}\right)$, $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ is in Q.

Lemma (2.1.3) Let α_i be k distinct nonzero numbers. If for some suitable m, the λ_i satisfy

$$\sum \lambda_i \alpha_i^r = 0 \quad \text{for} \quad m \leq r < m + k,$$

then all the λ_i are zero.

Proof. We can rewrite the equation as:

$$\sum\left(\lambda_i \alpha_i^m\right) \cdot \alpha_i^s = 0 \quad \text{for } 0 \leq s < k$$

and since the Vandermonde determinant $\det(\alpha_i^s)$ is non-zero, the solution for λ_i can only be zero. □

(2.1.4) Let $\mathcal{F}_0, \mathcal{G}_0$ be two semisimple $\overline{\mathbb{Q}}_\ell$-coefficients of rank r over X_0. Let $\alpha_x(\mathcal{F}_0, \mathcal{G}_0) := \alpha_x(\mathcal{F}_0 \oplus \mathcal{G}_0) = \sup\left(\alpha_x(\mathcal{F}_0), \alpha_x(\mathcal{G}_0)\right)$.

Proposition (2.1.5) If for every integer $n \leq N$ and every $x \in X_0(\mathbb{F}_{q^n})$, we have

$$\mathrm{Tr}(F_x, \mathcal{F}_0) = \mathrm{Tr}(F_x, \mathcal{G}_0)$$

then $\mathcal{F}_0$ is isomorphic to $\mathcal{G}_0$.

Proof. Applying (A.2.9) and (A.2.10) to $\mathcal{F}_0 \oplus \mathcal{G}_0$, we obtain decompositions:

$$\begin{aligned} \mathcal{F}_0 &= \bigoplus_{a \in A} p_{a*}\left(\mathcal{S}_{a,1} \otimes \mathrm{pr}_a^* \mathcal{W}_a\right) \\ \mathcal{G}_0 &= \bigoplus_{a \in A} p_{a*}\left(\mathcal{S}_{a,1} \otimes \mathrm{pr}_a^* \mathcal{W}'_a\right). \end{aligned} \tag{2.1}$$

In (2.1), for each $a \in A$, there is an integer $n(a) \geq 1$, such that $\mathcal{S}_{a,1}$ is a $\overline{\mathbb{Q}}_\ell$-coefficient on $X_a := X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n(a)}}$, $\mathcal{W}_a$ and $\mathcal{W}'_a$ are $\overline{\mathbb{Q}}_\ell$-Weil sheaves on $\mathrm{Spec}\left(\mathbb{F}_{q^{n(a)}}\right)$, p_a and pr_a are the projections from X_a to X_0 and $\mathrm{Spec}\left(\mathbb{F}_{q^{n(a)}}\right)$, respectively.

Let $\mathcal{S}_{a,1}^{(i)}$ denote the image of $\mathcal{S}_{a,1}$ under the i-th power of Frobenius $F \in \mathrm{Gal}\left(\mathbb{F}_{q^{n(a)}}/\mathbb{F}_q\right)$, and we omit the subscript 1 to indicate its inverse image of X_a to X.

According to (A.2.10), the $\mathcal{S}_{a,1}$, $\mathcal{W}_a$, and $\mathcal{W}'_a$ in (2.1) satisfy:

(i) $\det(\mathcal{S}_{a,1})$ is of finite order.

(ii) The $\mathcal{S}_a^{(i)}$ (where $a \in A$ and $i \in \mathbb{Z}/n(a)$) are irreducible $\overline{\mathbb{Q}}_\ell$-coefficients on X, pairwise non-isomorphic.

(iii) For each a, either $\mathcal{W}_a$ or $\mathcal{W}'_a$ is nonzero.

Since by definition $\mathcal{S}_a^{(i)}$ is a direct factor of either $\mathcal{F}$ or $\mathcal{G}$, we have

$$\alpha_x\left(\mathcal{S}_a^{(i)}\right) \leq \sup\left(\alpha_x(\mathcal{F}), \alpha_x(\mathcal{G})\right)$$

Let $A(n)$ denote the set of a in A such that $n(a)|\, n$.

Key Claim (2.1.6) If $n > N_0$ (2.1.1), the functions on $X_0(\mathbb{F}_{q^n})$

$$t_{a,i} : x \longmapsto \mathrm{Tr}\left(F_x, \mathcal{S}_{a,1}^{(i)}\right) \quad (a \in A(n), i \in \mathbb{Z}/n(a))$$

are linearly independent.

Proof of Claim: The functions $t_{a,i}$ take values in a number field $E \subset \overline{\mathbb{Q}}_\ell$. Let us embed E into $\mathbb{C}$ to treat them as functions with complex values. The idea of the proof is to show that they are almost orthogonal, in $L^2(X_0(\mathbb{F}_{q^n}))$.

According to [Laf02, VII 6 (i)], $\mathcal{S}_{a,1}^{(i)}$ is pure of weight 0. The complex conjugate of $t_{a,i}$ is thus given by $x \mapsto \mathrm{Tr}\left(F_x, \mathcal{S}_{a,1}^{(i)\vee}\right)$, and the inner product $\langle t_{b,j}, t_{a,i}\rangle = \sum t_{a,i}(x)^- t_{b,j}(x)$ is

$$\langle t_{b,j}, t_{a,i}\rangle = \sum \mathrm{Tr}\left(F_x, \mathscr{H}\!om\left(\mathcal{S}_{a,1}^{(i)}, \mathcal{S}_{b,1}^{(j)}\right)\right).$$

By the trace formula, this sum is the trace of the Frobenius $F \in W(\mathbb{F}/\mathbb{F}_{q^n})$ on the compact cohomology:

$$\langle t_{b,j}, t_{a,i}\rangle = \sum (-1)^k \,\mathrm{Tr}\left(F, H_c^k\left(X, \mathscr{H}\!om\left(\mathcal{S}_a^{(i)}, \mathcal{S}_b^{(j)}\right)\right)\right).$$

Since X is affine, H_c^0 is trivial. The "dominant" term is given by H_c^2: it equals q^n if $(a,i) = (b,j)$ and is zero otherwise. Precisely, $H_c^2\left(X, \mathscr{H}\!om\left(\mathcal{S}_b^j, \mathcal{S}_a^i\right)\right) \simeq \mathrm{Hom}\left(\mathcal{S}_a^i\big|_X, \mathcal{S}_b^j\big|_X\right)^\vee(-1)$.

The promised almost orthogonality comes from the fact that the $k = 1$ term is of order $O\left(q^{n/2}\right)$ for large n. More precisely, since $\mathscr{H}\!om\left(\mathcal{S}_{a1}^{(i)}, \mathcal{S}_{b1}^{(j)}\right)$ is pure of weight 0, the eigenvalues of F on its H_c^1 are of absolute value $q^{n/2}$ or 1 by Weil II, using (B.2), the $k=1$ term is bounded in absolute value by $q^{n/2}$ times

$$\dim H_c^1 \leq \dim \mathcal{S}_b^{(j)} \cdot \dim \mathcal{S}_a^{(i)} \cdot \left(b_1(X) + \sum \alpha_x(\mathcal{F}, \mathcal{G})\right). \tag{2.2}$$

Suppose there exists a linear dependence relation $\sum \lambda_{b,j} t_{b,j} = 0$. Let a, i be such that $\left|\lambda_{a,i}\right|$ is maximal among all $\left|\lambda_{b,j}\right|$. Dividing by $\lambda_{a,i}$, we can assume that $\lambda_{a,i} = 1$ and $\left|\lambda_{b,j}\right| \leq 1$ for $b \in A(n)$ and $j \in \mathbb{Z}/n(b)$. We then have the following:

$$\begin{aligned} 0 = \left\langle \sum \lambda_{b,j} t_{b,j}, t_{a,i} \right\rangle &= \sum_{b,j} \lambda_{b,j} \langle t_{b,j}, t_{a,i} \rangle \\ &= q^n + \text{ remainder}, \end{aligned} \tag{2.3}$$

where the absolute value of the remainder is bounded by

$$\begin{aligned} &\sum_{b,j} \left|\lambda_{b,j}\right| \operatorname{Tr}\left(F, H_c^1\left(X, \mathscr{H}om\left(\mathcal{S}_a^{(i)}, \mathcal{S}_b^{(j)}\right)\right)\right) \\ &\qquad \leq q^{n/2} \cdot 2r^2 \left(b_1(X) + \sum \alpha_x(\mathcal{F}, \mathcal{G})\right). \end{aligned}$$

The assumption on n ensures that $|\text{remainder}| < q^n$, leading to a contradiction.

□

Corollary (2.1.7) Let F be the Frobenius of $W(k/\mathbb{F}_{q^n})$. If $n > N_0$ and $\operatorname{Tr}(F_x, \mathcal{F}_0) = \operatorname{Tr}(F_x, \mathcal{G}_0)$ for $x \in X_0(\mathbb{F}_{q^n})$, then

$$\operatorname{Tr}(F, \mathcal{W}_a) = \operatorname{Tr}(F, \mathcal{W}'_a)$$

for every $a \in A(n)$.

Actually, the decompositions (2.1) yield the identity between trace functions:

$$\sum t_{a,i} \operatorname{Tr}(F, \mathcal{W}_a) = \sum t_{a,i} \operatorname{Tr}(F, \mathcal{W}'_a)$$

and we apply Lemma (2.1.6).

Let's go back to (2.1.5). We need to show that for each a, and for F the geometric Frobenius of $W(k/\mathbb{F}_{q^{n(a)}})$, F has the same multiset of eigenvalues on $\mathcal{W}_a$ and $\mathcal{W}'_a$. According to the claim and the assumption of proposition (2.1.5), if n is divisible by $n(a)$ and

$$\lfloor N_0 \rfloor + 1 \leq n \leq N = \lfloor N_0 \rfloor + 2r,$$

then we have

$$\operatorname{Tr}\left(F^{n/n(a)}, \mathcal{W}_a\right) = \operatorname{Tr}\left(F^{n/n(a)}, \mathcal{W}'_a\right).$$

There are at least $\lfloor 2r/n(a) \rfloor$ such values of n, and $\mathcal{W}_a$ and $\mathcal{W}'_a$ have dimension at most $\lfloor r/n(a) \rfloor$. It remains to apply the lemma (2.1.3) to the set of all eigenvalues of F on $\mathcal{W}_a$ and $\mathcal{W}'_a$. More precisely, if the set of all eigenvalues of F on $\mathcal{W}_a$ (resp. $\mathcal{W}'_a$) is $\{\alpha_i\}$ (resp. $\{\beta_j\}$), then $\sum \alpha_i - \sum \beta_j = 0$, by lemma (2.1.3) there must exist some $\alpha_i = \beta_j$, then we can make induction. □

(2.1.8) *Proof of Proposition (2.1.2).*

By semisimplifying $\mathcal{C}_0$, we can assume that $\mathcal{C}_0$ is semi-simple, so that we can apply (2.1.5). Let E be an appropriately large Galois extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ containing all $\operatorname{Tr}(F_x, \mathcal{C}_0)$ for any x in $X_0(\mathbb{F}_{q^n})$. To show that all trace contained in E must contained in Q, we need to show that for $\sigma \in \operatorname{Gal}(E/Q)$, the $\operatorname{Tr}(F_x, \mathcal{C}_0)$ are fixed by σ. This is equivalent to

$$\operatorname{Tr}(F_x, \mathcal{C}_0) = \operatorname{Tr}(F_x, \sigma(\mathcal{C}_0)). \tag{2.4}$$

By assumption, (2.4) holds if x is in $X_0(\mathbb{F}_{q^n})$ with $n \leq N$. Using (2.1.5) and the fact that $\alpha_x(\mathcal{C}_0) = \alpha_x(\sigma(\mathcal{C}_0))$[Del12, (2.2.1)], we conclude that $\mathcal{C}_0$ is isomorphic to $\sigma(\mathcal{C}_0)$. The assertion follows from this.

Variant (2.1.9) Let's keep the assumptions and notations from (2.1.5). Suppose $q : X' \to X$ is a connected étale covering of X on which the inverse images of $\mathcal{F}$ and $\mathcal{G}$ have tame ramification. For any $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{H}$ on X, whose inverse image on X' has tame ramification, the morphism

$$q : H^*(X, \mathcal{H}) \to H^*(X', q^*\mathcal{H}).$$

is injective, since its composition with Tr_q is multiplication by the degree of the covering. Therefore, we have

$$\dim H^1_c(X, \mathcal{H}) \leq \dim H^1_c(X', \mathcal{H}) \leq \mathrm{rank}(\mathcal{H}) \cdot b_1(X') \tag{2.5}$$

If we repeat the arguments that prove (2.1.5) using this estimate instead of (2.5), we obtain:

Variant (2.1.10) Let $N'_0 := 2\log_q^+ \left(2r^2 b_1(X')\right)$ and $N' := \lfloor N'_0 \rfloor + 2r$. If for every integer $n \leq N'$ and every $x \in X_0(\mathbb{F}_{q^n})$, we have

$$\mathrm{Tr}(F_x, \mathcal{F}_0) = \mathrm{Tr}(F_x, \mathcal{G}_0),$$

then $\mathcal{F}_0$ is isomorphic to $\mathcal{G}_0$.

Similarly, if $\mathcal{F}_0$ is as in (A.1.12), and its inverse image under $q : X' \to X$ has tame ramification, then $\sigma(\mathcal{F}_0)$ has the same property: after a finite extension of the base field $\mathbb{F}_q$, we can assume that $q : X' \to X$ arises from $q_0 : X'_0 \to X_0$, and we apply the fact that if $\mathcal{F}$ is tamely ramified, then $\sigma(\mathcal{F})$ is also tamely ramified (this follows from the Grothendieck trace formula (A.1.10) and (B.2.6)) . By repeating the same arguments that prove (A.1.12), we obtain the following:

Variant (2.1.11) Let Q be the extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ generated by the $\mathrm{Tr}(F_x, \mathcal{F}_0)$ for x in $X_0(\mathbb{F}_{q^n})$ with $n \leq N'$, where N' is as in (2.1.10). Then, for every n and every $x \in X_0(\mathbb{F}_{q^n})$, $\mathrm{Tr}(F_x, \mathcal{F}_0)$ is in Q.

2.2 General Case

(2.2.1) For the proof of the theorem, we can reduce to the case that

(i) X_0 is an affine, smooth, irreducible scheme equipped with an étale morphism $\varphi : X_0 \to \mathbb{A}^k_0$ to an affine space over $\mathbb{F}_q$;

(ii) $\mathcal{C}_0$ is lisse, irreducible, and $\det(\mathcal{C}_0)$ is of finite order.

Actually the scheme X_0 in the theorem admits a partition into locally closed irreducible parts F_i satisfying: (i) and such that $\mathcal{C}_0|_{F_i}$ is lisse. It suffices to treat each $\left(F_i, \mathcal{C}_0|_{F_i}\right)$ separately. We can then treat each irreducible subquotient of $\mathcal{C}_0|_{F_i}$ separately and twist it to satisfy (ii).

Now let's state the main proposition who implies the main theorem (2.0.1).

Proposition (2.2.2) If the integer n is sufficiently large, for any $x \in X_0(\mathbb{F}_{q^n})$, the trace $\mathrm{Tr}(F_x, \mathcal{C}_0)$ is contained in the extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$ generated by the traces $\mathrm{Tr}(F_y, \mathcal{C}_0)$ with $y \in X_0(\mathbb{F}_{q^m})$ and $m < n$.

If this proposition holds, then N is chosen such that for all integers $n > N$, the condition of being "sufficiently large" holds, then we can deduce by induction that every trace $\mathrm{Tr}(F_x, \mathcal{C})$ is contained in the extension of $\mathbb{Q}$ generated by the traces $\mathrm{Tr}(F_y, \mathcal{C}_0)$ with y in $X_0(\mathbb{F}_{q^m})$ for $m \leq N$. This extension is finite, therefore giving the main theorem (2.0.1).

(2.2.3) Let's choose a generator of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. This choice defines an $\mathbb{F}_{q^n}$-point x' of the affine line $\mathbb{A}^1_0$ over $\mathbb{F}_q$, whose Galois conjugates under $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ are all distinct.

Lemma (2.2.4) If y is an $\mathbb{F}_{q^n}$-point of $\mathbb{A}^1_0$, there exists an $\mathbb{F}_q$-morphism $P : \mathbb{A}^1_0 \to \mathbb{A}^1_0$, i.e., a polynomial $P \in \mathbb{F}_q[T]$, that sends x' to y and has degree $\leq n-1$.

Proof. The $\sigma x'$, for $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$, are n distinct points on the affine line. Hence, there exists a unique polynomial $P \in \mathbb{F}_{q^n}[T]$, of degree $\leq n-1$, such that for each σ, it takes the value σy at $\sigma x'$, which implies that P sends x' to y. Its uniqueness guarantees its invariance under Galois, which implies that it belongs to $\mathbb{F}_q[T]$. □

Corollary (2.2.5) There exists an $\mathbb{F}_q$-morphism $P : \mathbb{A}^1_0 \to \mathbb{A}^k_0$, with coordinate polynomials of degree $\leq n-1$, that sends x' in $\mathbb{A}^1_0(\mathbb{F}_{q^n})$ to $\varphi(x)$ in $\mathbb{A}^k_0(\mathbb{F}_{q^n})$.

Proof. This follows by applying Lemma (2.2.4) to each coordinate of $\varphi(x)$. □

(2.2.6) Let us fix P as in (2.2.5). Consider X''_0 as the fiber product of X_0 and $\mathbb{A}^1_0$ over $\mathbb{A}^k_0$. Since $\varphi(x) = P(x')$, there exists $\bar{x} \in X''_0(\mathbb{F}_{q^n})$ that maps to both x and x'. We denote by Z_0 the connected component of X''_0 containing $\bar{x}$ (which is a curve):

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[dr, "\widetilde{P}"'] & X''_0 \arrow[r, "\varphi''"] \arrow[d] & \mathbb{A}^1_0 \arrow[d, "P"] & \bar{x} \in Z_0\,(\mathbb{F}_{q^n}) \arrow[d] \\ & X_0 \arrow[r, "\varphi"] & \mathbb{A}^k_0 & x \in X_0\,(\mathbb{F}_{q^n}) \end{tikzcd} \tag{2.6}$$

The morphism φ'' is étale (since φ is étale and use base change), and its degree over the generic point of $\mathbb{A}^1_0$ is at most equal to the degree D of φ. Let k_0 be the field of constants of the curve Z_0 over $\mathbb{F}_q$. The degree d of k_0 over $\mathbb{F}_q$ is less than or equal to D because Z_0 has degree at most D over $\mathbb{A}^1_0$, and d divides n since Z_0 has a point over $\mathbb{F}_{q^n}$. We extend the base field from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, and let Z_1 (resp. X_1) be the connected component of $Z_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d}$ (resp. $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^d}$) containing $\bar{x}$ (resp. x):

$$\begin{tikzcd} \bar{x} : \mathrm{Spec}\,(\mathbb{F}_{q^n}) \arrow[r] \arrow[d, "="] & Z_1 \arrow[r, "\sim"] \arrow[d] & Z_0 \arrow[d, "\widetilde{P}"] \\ x : \mathrm{Spec}\,(\mathbb{F}_{q^n}) \arrow[r] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ & \mathrm{Spec}\,(\mathbb{F}_{q^d}) \arrow[r] & \mathrm{Spec}\,(\mathbb{F}_q) \end{tikzcd} \tag{2.7}$$

The curve Z_1 over $\mathbb{F}_{q^d}$ is smooth and absolutely irreducible (cf. [Stacks, Tag 04KZ]).

Finally, we extend the base field from $\mathbb{F}_{q^d}$ to $\mathbb{F} = \overline{\mathbb{F}}_q$ to obtain X and Z. For X, there exists a connected étale covering $q : X' \to X$ such that the monodromy group of $q^*\mathcal{C}$ is a pro-ℓ-group (for the proof, see (3.2.4)). Let Z' be a connected component of the inverse image Z'' of the étale covering X' of X:

$$\begin{tikzcd} Z' \arrow[r, hook] & Z'' \arrow[r] \arrow[d] & Z \arrow[d] \\ & X' \arrow[r] & X \end{tikzcd}$$

(2.2.7) *Proof of Proposition (2.2.2).*

The inverse image of $\mathcal{C}$ on Z' is tamely ramified. We aim to apply (2.1.11) to the curve Z_1 over $\mathbb{F}_{q^d}$, to $Z' \to Z$, and to the inverse image $\mathcal{C}_1$ of $\mathcal{C}$ on Z_1. Let's estimate $b_1(Z')$.

The composed $\mathbb{F}$-morphism

$$X' \to X \xrightarrow{\varphi} \mathbb{A}^k$$

is affine. Therefore, it factors through a closed embedding $X' \hookrightarrow \mathbb{A}^{k+k'}$, and there exists a family of A' equations of degree $\le b$ defining X' in $\mathbb{A}^{k+k'}$.

The graph Γ_P of $P : \mathbb{A}^1 \to \mathbb{A}^k$ is defined in $\mathbb{A}^{1+k}$ by k equations of degree $\le n-1$. The fiber product, computed over $\mathbb{F}$, of $X' \to \mathbb{A}^k$ and $P : \mathbb{A}^1 \to \mathbb{A}^k$ is the intersection, in $\mathbb{A}^{1+k+k'}$, of the inverse images of $X' \subset \mathbb{A}^{k+k'}$ and $\Gamma_P \subset \mathbb{A}^{1+k}$. Therefore, it is defined in $\mathbb{A}^{1+k+k'}$ by $A := A' + k$ equations of degree $\le \sup(B, n-1)$.

According to Katz [Kat01], this yields an upper bound on the sum of Betti numbers of this fiber product, and hence on $b^1(Z')$, since Z' is a connected component of the fiber product.

For $n > B$, we have A equations of degree $\le n-1$ in an affine space of dimension $k + k' + 1$, and Katz gives

$$b_1\left(Z'\right) \le 6.2^A (An+3)^{k+k'+1}. \tag{2.8}$$

Let us define, as before,

$$N'_0 = 2\log^+_{q^d}\left(2r^2 b_1\left(Z'\right)\right) \quad \text{and}$$
$$N' = \lfloor N'_0 \rfloor + 2r.$$

The bound (2.8) is polynomial in n, and d is bounded by D. As soon as n is sufficiently large, we have

$$\frac{n}{d} > N'.$$

Let us assume that n is large enough for this inequality to hold. From (2.7) , we have $\mathrm{Tr}\left(F_{\bar{x}}, \mathcal{C}_1\right) = \mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$. According to (2.1.11), this trace is contained in the field generated by $\mathrm{Tr}\left(F_y, \mathcal{C}_1\right)$ for y in $Z_1\left(\mathbb{F}_{q^m}\right)$ with $d | m$ and $\frac{m}{d}$ (the degree of $\mathbb{F}_{q^m}$ over $\mathbb{F}_{q^d}$) at most equal to N'. Moreover, by (2.7), these traces are also $\mathrm{Tr}\left(F_y, \widetilde{P}^*\mathcal{C}_0\right)$ for y in $X_0\left(\mathbb{F}_{q^m}\right)$ with $\frac{m}{d} \le N'$. We have

$$\mathrm{Tr}\left(F_y, \widetilde{P}^*\mathcal{C}_0\right) = \mathrm{Tr}\left(F_{\widetilde{P}(y)}, \mathcal{C}_0\right).$$

And $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ belongs to the field generated by $\mathrm{Tr}\left(F_y, \mathcal{C}_0\right)$ with $y \in X_0\left(\mathbb{F}_{q^m}\right)$ and $\frac{m}{d} \le N' < \frac{n}{d}$, and thus $m < n$. This proves (2.2.2).

2.3 An Alternative Proof [EK11] (By H. Esnault and M. Kerz)

(2.3.1) In [EK11] H. Esnault and M. Kerz use ramification theory to give an alternative proof which is a bit more direct than N. Katz's estimation of Betti number. By (1.3.4) (3) , it's enough to consider the case $\overline{\mathbb{Q}}_\ell$-coefficient pure of weight 0.

(2.3.2) Let $X'_0 \to X_0$ be a finite dominant morphism with X_0 normal noetherian integral and X'_0 integral. And $K \subset K'$ is the corresponding extension of the fields of rational functions. Consider the diagonal morphism $\phi : X'_0 \to X'_0 \times_{X_0} X'_0$. Let $\mathcal{I} \subset \mathcal{O}_{X'_0 \times_{X_0} X'_0}$ be the coherent ideal sheaf of the diagonal.

Definition (2.3.3) The homological different of X'_0 over X_0 is defined as the coherent ideal sheaf

$$\mathrm{Diff}_{X'_0/X_0} = \phi^*\left(\mathrm{Ann}_{\mathcal{O}_{X'_0 \times_{X_0} X'_0}}(\mathcal{I})\right) \subset \mathcal{O}_{X'_0}.$$

Here $\phi^\natural$ is the usual pullback of ideal sheaves. Taking norms we get the coherent ideal sheaf

$$\mathrm{D}_{X'_0/X_0} = \mathcal{O}_{X_0}\mathrm{Nm}_{K'/K}\left(\mathrm{Diff}_{X'_0/X_0}\right) \subset \mathcal{O}_{X_0}.$$

(2.3.4) Let $\overline{X}_0 \supset X_0$ be an open immersion with $\overline{X}_0$ integral, normal, proper over $\mathbb{F}_q$. Let $D \in \mathrm{Div}^+(\overline{X}_0)$ be an effective Cartier divisor on $\overline{X}_0$ which is supported in $\overline{X}_0 \backslash X_0$.

Definition (2.3.5) we define the complexity of D to be

$$\mathcal{C}_D = 2g(\bar{X}_0) + 2\deg(D) + 1.$$

Definition (2.3.6) Given a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$, we say that $\mathcal{C}_0$ is tame if its pullback along any curve $C \to X_0$ is tame, see (B.2.2).

We say that the (wild) ramification of $\mathcal{C}_0$ is bounded by D if there is a connected étale covering $\phi : X_0' \to X_0$ such that $\phi^*(\mathcal{C}_0)$ is tame and such that $\mathcal{O}_{\overline{X}_0}(-D) \subset \mathrm{D}_{\bar{X}_0'/\overline{X}_0}$, where $\overline{X}_0'$ is the normalization of $\overline{X}_0$ in $k(X_0')$.

(2.3.7) Consider an nonempty open subscheme $X_0 \subset \mathbb{P}^d_{\mathbb{F}_q}$ and an effective Cartier divisor $D_{\mathbb{F}} \in \mathrm{Div}^+\left(\mathbb{P}^d_{\mathbb{F}}\right)$ $(\mathbb{F} = \overline{\mathbb{F}}_q)$ with support equal to $(\mathbb{P}^d_{\mathbb{F}_q} \backslash X)_{\mathbb{F}}$.

Lemma (2.3.8) Let $\mathcal{C}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient on X_0 of rank r which is pure of weight 0 and with ramification of $\mathcal{C}$ bounded by D. Let Q be the number field generated by the coefficients of $\chi_x(\mathcal{C}, T)$ for $x \in |X_0|$ with

$$\deg(x) \le 4r^2 \left\lfloor \log_q \left(8r^2 \deg(x) \deg(D) + 4r^2\right)\right\rfloor.$$

Then for every n and every $x \in X_0\left(\mathbb{F}_{q^n}\right)$, $\mathrm{Tr}\left(F_x, \mathcal{C}_0\right)$ is in Q.

Proof. We prove the lemma by induction on n that for $x \in |X_0|$ with $\deg(x) \le n$, $\chi_x(\mathcal{C}, T) \in Q[t]$. Consider a point x with $\deg(x) = n$ such that

$$n \ge 4r^2 \left\lfloor \log_q \left(8r^2 n \deg(D) + 4r^2\right)\right\rfloor.$$

Let $x \in \mathbb{A}^d_{\mathbb{F}_q} \subset \mathbb{P}^d_{\mathbb{F}_q}$ be an open subscheme with

$$\mathbb{A}^d_{\mathbb{F}_q} = \mathrm{Spec}\left(\mathbb{F}_q\left[T_1, \ldots, T_d\right]\right).$$

The point x gives rise to a homomorphism $\mathbb{F}_q\left[T_1, \ldots, T_d\right] \to \mathbb{F}_{q^n}$. We choose an embedding $x \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q} = \mathrm{Spec}\left(\mathbb{F}_q[T]\right)$ and a lifting

$$\phi : \mathbb{F}_q\left[T_1, \ldots, T_d\right] \to \mathbb{F}_q[T]$$

with $\deg\left(\phi\left(T_i\right)\right) < n$ $(1 \le i \le d)$ as in (2.2.5). By projective completion we get a morphism $\psi : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^d_{\mathbb{F}_q}$ of degree less than n extending the map $x \to X_0$.

Consider the curve $C = \psi^{-1}(X_0)$ and the divisor $D_C = \psi^*(D)$ on $\mathbb{P}^1_{\mathbb{F}}$. By [EK12, Proposition 4.8] the ramification of the sheaf $\psi^*(\mathcal{C})$ is bounded by D_C. Clearly $\mathcal{C}_{D_C} \le 2n\deg(D) + 1$ by our assumption on n. By [EK12, Theorem 5.6] the coefficients of $f_{\psi^*\mathcal{C}_0}(x)$ are contained in the field generated by the coefficients of the $f_{\psi^*\mathcal{C}_0}(z)$ with $z \in C$ and

$$\deg(z) \le 4r^2 \left\lfloor \log_q \left(4r^2 \mathcal{C}_{D_C}\right)\right\rfloor.$$

The latter coefficients are contained in Q by induction, since we have

$$4r^2 \left\lfloor \log_q \left(4r^2 \mathcal{C}_{D_C}\right)\right\rfloor < n$$

by our assumption on $n = \deg(x)$. ∎

(2.3.9) For general scheme, since X_0 is normal, we can assume it is integral by taking a connected component. By Noetherian induction, we may replace X_0 by a dense open subscheme. Thus by using Noether Normalization (cf. [Stacks, Tag 0CBL]) we can assume that there is a closed immersion $X_0 \hookrightarrow \mathbb{A}^1 \times_{\mathbb{F}_q} Y$ with Y an open subscheme of $\mathbb{A}^d_{\mathbb{F}_q}$ such that $X_0 \to Y$ is finite étale.

W take $\mathcal{C}_0$ as an object of $D^b_c\left(\mathbb{A}^1 \times_{\mathbb{F}_q} Y\right)$, concentrated in degree 0. We fix a nontrivial character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ and we let $\mathcal{F}(\mathcal{C}_0) \in D^b_c\left(\mathbb{A}^1 \times_{\mathbb{F}_q} Y\right)$ be the corresponding Fourier-Deligne transformation of $i_*\mathcal{C}_0$ over the base Y, then $\mathcal{F}(\mathcal{C}_0)$ is concentrated in degree -1. By (2.3.8) we get a the trace field $Q_{\mathcal{F}(\mathcal{C}_0)}$ for $\mathcal{C}_0$. Then using trace formula and the Fourier inversion formula [BBD82] we can prove (1.3.3) also holds for $\mathcal{C}_0$.

3 (2)-Skeleton Sheaves

In this section, let X denote a normal, geometrically connected scheme of finite type over $\mathbb{F}_q$. Let Q be an algebraic extension of $\mathbb{Q}_\ell$ with ring of integers Z and $\mathcal{C}$ is a Q-coefficient. The idea of associating a Q-coefficient with its family of restrictions to curves on X can be formalized by introducing the notion of a Q-skeleton, the terminology is introduced in [EK12].

3.1 Skeleton

(3.1.1) The set $\mathcal{C}_{Q,r}(X)$. Say that two given Q-coefficients on X are equivalent if their semi-simplifications coincide. Let $\mathcal{C}_{Q,r}(X)$ be the set of equivalence classes of Q-coefficients on X of rank r. Obviously, $\mathcal{C}_{Q,r}(-)$ is a contravariant functor.

Recall that in (A.1.13), for a given Q-coefficients $\mathcal{C}$ we can define a map: $|X| \to \mathcal{P}_r\left(\overline{\mathbb{Q}}_\ell\right)$. Let $\mathcal{L}_r(X)$ be the product $\prod_{|X|} \mathcal{P}_r$ with one copy of $\mathcal{P}_r$ for every closed point of X. We define a map from Q-coefficients to $\mathcal{L}_r(X)$, obviously it factors through $\mathcal{C}_{Q,r}(X)$ since characteristic polynomial is determined up to the semi-simplification.

If X_{red} is normal this map is injective by the Čebotarev density theorem. Briefly, Čebotarev density theorem tell us that in a finite quotient of the Galois group, every element arises from a Frobenius conjugacy class, and if the identity on every Frobenius element passing to every elements, and use a general fact about representations of groups over fields of characteristic zero, we can identify lisse sheaves (up to semi-simplification) with the map defined above.

(3.1.2) Let us denote $\mathrm{Cu}(X)$ as the set of pairs (C, ϕ), where C is a smooth curve over k and $\phi : C \to X$ is a k-morphism. A Q-skeleton of rank r is defined as an element of the equalizer $\mathcal{S}_{Q,r}(X)$ defined by the diagram below, where the arrows represent the arrows, which are composed of restriction and semi-simplification arrows.

$$\begin{tikzcd}
\mathcal{S}_{Q,r}(X) \arrow[r] & \prod_{C\in\mathrm{Cu}(X)} \mathcal{C}_{Q,r}(C) \arrow[r, shift left] \arrow[r, shift right] & \prod_{C,C'\in\mathrm{Cu}(X)} \mathcal{C}_{Q,r}\left((C\times_X C')_{red}\right) \\
& \mathcal{C}_{Q,r}(X,Q) \arrow[u] \arrow[ul, dashed, "\mathrm{Sq}"] &
\end{tikzcd}$$

(3.1.3) For every $x \in |X|$, the map $\chi_x : \mathcal{C}_{Q,r}(X) \to \mathcal{L}_r(X)$ defined by sending $x \in |X|$ to $\chi_x(\mathcal{C})$ extends to $\mathcal{S}_{Q,r}(X)$ as follows. For each $x \in |X|$, we choose a curve $C_x \in \mathrm{Cu}(X)$ such that $C_x \to X$ is a closed immersion near x. The map $\chi_x : \mathcal{S}_{Q,r}(X) \to \mathcal{L}_r(X)$ defined by $\chi_x(\mathcal{C}) = \chi_x(\mathcal{C}_{C_x})$, where $\mathcal{C} = (\mathcal{C}_C)_{C\in\mathrm{Cu}(X)} \in \mathcal{S}_{Q,r}(X)$, is well-defined, injective, and satisfies $\chi_x \circ \mathrm{Sq} = \chi_x$. In particular, according to the Čebotarev theorem, the map $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}_{Q,r}(X)$ is injective.

(3.1.4) The trace field $Q_{\mathcal{C}}$ of a Q-skeleton $\mathcal{C}$ is defined as the sub-$\mathbb{Q}$-extension of Q generated by the coefficients of $\chi_x(\mathcal{C}, T)$, $x \in |X|$ (equivalently, by $Q_{\mathcal{C}_C}$ for $C \in \mathrm{Cu}(X)$). We say that a Q-skeleton $\mathcal{C}$ is ι-pure of weight w (respectively, pure of weight w, algebraic) if $\mathcal{C}_C$ for $C \in \mathrm{Cu}(X)$ satisfy the corresponding properties. Furthermore, the direct sum of two skeletons is defined in an obvious manner, and a skeleton is said to be irreducible if it cannot be written as the direct sum of two non-zero skeletons.

(3.1.5) We say that a Q-skeleton $\mathcal{C}$ is tame if the $\mathcal{C}_C$, for $C \in \mathrm{Cu}(X)$, are tame in the usual sense of curve, see (B.2.2). We define the pullback $f^*\mathcal{C}$ of a Q-skeleton $\mathcal{C}$ by a k-morphism $f\colon Y \to X$ as follows:

$$(f^*\mathcal{C})_{(C,\phi)} = \mathcal{C}_{(C,f\circ\phi)}, (C, \phi \in \mathrm{Cu}(Y))$$

We say that $\mathcal{C}$ is tame along $f\colon Y \to X$ if $f^*\mathcal{C}$ is tame.

3.2 Geometric Skeleton

Definition (3.2.1)

1. An alteration of X is a morphism $f\colon X' \to X$ which is proper, surjective, and generically finite étale.

2. A smooth pair over k is a pair (Y, Z) in which Y is a smooth k-scheme and Z is a strict normal crossings divisor on Y; we refer to Z as the boundary of the pair. Note that $Z = \emptyset$ is allowed.

3. A good compactification of X is a smooth pair $(\overline{X}, Z)$ over k with $\overline{X}$ projective (not just proper) over k, together with an isomorphism $X \cong \overline{X} \backslash Z$.

4. Let $X \hookrightarrow \overline{X}$ be an open immersion with dense image. Let D be an irreducible divisor of $\overline{X}$ with generic point η. Let $\mathcal{C}$ be a $\overline{\mathbb{Q}}_\ell$-coefficient on X. We say that $\mathcal{C}$ is docile along D if the action of the inertia group of η on $\mathcal{C}$ tamely ramified and unipotent.

(3.2.2) If $\overline{X}$ is a good compactification of X with boundary Z, then a $\overline{\mathbb{Q}}_\ell$-coefficient on X is tame (resp. docile) if and only if it is so with respect to each component of Z. Namely, this follows from Zariski-Nagata purity.

Proposition (3.2.3) ([Jon96], Theorem 4.1). There exists an alteration $f\colon X' \to X$ such that X' admits a good compactification. (Beware that X' is not guaranteed to be geometrically irreducible over k.)

Proposition (3.2.4) For any $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}$ on X, there exists an alteration $f\colon X' \to X$ such that X' admits a good compactification and $f^*\mathcal{E}$ is docile.

Proof. Indeed, $\mathcal{C}$ arises from a scalar extension of a smooth Z_λ-sheaf $\mathcal{H}_\lambda$, where Z_λ is the ring of integers of a finite extension Q_λ of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$. By considering λ as the uniformizer of Z_λ, we can take the étale cover trivializing $\mathcal{H}_\lambda/\lambda$, i.e. the étale covering associating to the kernel of the representation (which is normal and open). Then use the fact that the kernel of the homomorphism $\mathrm{GL}(r, Z_\lambda) \to \mathrm{GL}(r, Z_\lambda/\lambda)$ is a pro-ℓ-group, so it cannot contain nontrivial pro-p-subgroups for $p \neq \ell$. So we prove it's tame.

To prove the action is unipotent. We prove it's quasi-unipotent by the usual argument of Grothendieck: the eigenvalues of Frobenius form a multiset of length at most $r := \mathrm{rank}(\mathcal{C})$ which is stable under taking p-th powers, so this multiset must consist entirely of roots of unity. To upgrade from quasi-unipotence to unipotence, it suffices to further trivialize the action by consider $\mathrm{GL}(r, Z_\lambda/\lambda^m)$ for some m. $\square$

Remark (3.2.5) The proof above also shows that any $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}$ on X is tame by a connected étale cover.

(3.2.6) Recall that a pro-finite group is said to be almost pro-ℓ if it has an open pro-ℓ-subgroup, and in this case the open pro-ℓ-subgroup can be chosen to be normal by conjugation. A topological group is termed topologically finitely generated if there exists a dense finitely generated subgroup.

The Galois description even tells us that if Q_λ is a finite extension of $\mathbb{Q}_\ell$, for any $\mathcal{C} \in \mathcal{C}_{Q_\lambda}(X)$, there exists a family of Galois covers $X_n \to X, n \geq 1$ (the étale covers trivializing $\mathcal{H}_\lambda / \lambda^n$) such that

(1) For all $x \in |X_n|, \chi_x(\mathcal{C}, T) \equiv (1-T)^r \left[\lambda^n\right], n \geq 1$;

(2)$\pi_1(X)/\Pi$ is an almost pro-ℓ-group, topologically of finite type, where $\Pi := \bigcap_{n\geq 1} \pi_1(X_n)$.

Lemma (3.2.7) Let $\mathcal{E}$ and $\mathcal{F}$ be $\overline{\mathbb{Q}}_\ell$-coefficients on X which are companions, then $\mathcal{E}$ is tame (resp. docile) if and only if $\mathcal{F}$ is.

Proof. For any $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{E}$ on X, we have (B.2.6)

$$\chi_c(\mathcal{E}) = \chi_c(X)\operatorname{rank}(\mathcal{E}) - \sum_{x\in \overline{X}\setminus X} n(x)\operatorname{Swan}_x(\mathcal{E}),$$

and $\mathcal{E}$ is tame iff $\sum_{x\in\overline{X}\setminus X} n(x)\operatorname{Swan}_x(\mathcal{E}) = 0$, but by (A.1.10) $\chi_c(\mathcal{C})$ is the order of vanishing of $L(\mathcal{E}^\vee, T)$ at $T=\infty$. ☐

(3.2.8) Therefore, we will say that a Q-skeleton $\mathcal{C}$ is 1-geometric if for every subscheme $Y \subset X$, there exists an alteration $Y' \to Y$ that $\mathcal{C}$ is tame along $Y' \to Y$, and we will denote $\mathcal{S}^{1-\mathrm{geom}}_{Q,r}(X) \subset \mathcal{S}_{Q,r}(X)$ as the corresponding subset. The canonical restriction map $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}_{Q,r}(X)$ therefore factors through $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}^{1-\mathrm{geom}}_{Q,r}(X)$.

(3.2.9) We will say that a Q-skeleton is geometric if it satisfies (3.2.6) for a finite extension Q_λ of $\mathbb{Q}_\ell$ inside Q, and we will denote $\mathcal{S}^{\mathrm{geom}}_{Q,r}(X) \subset \mathcal{S}^{1-\mathrm{geom}}_{Q,r}(X)$ as the corresponding subset. The canonical restriction map $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}_{Q,r}(X)$ therefore factors through $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}^{\mathrm{geom}}_{Q,r}(X)$.

Theorem (3.2.10) For any algebraic $\mathcal{C} \in \mathcal{S}^{1-\mathrm{geom}}_{\overline{\mathbb{Q}}_\ell,r}(X)$, the trace field $Q_{\mathcal{C}}$ is a finite extension of $\mathbb{Q}$.

Proof. See [Del12, Remark 3.10]. This alternatively from Theorem (4.1.1). ☐

4 V. Drinfeld's Main Theorem

4.1 Structure of the Proof

Theorem (4.1.1) ([Dri18, Theorem 2.5])

Let Q be a finite extension of $\mathbb{Q}_\ell$. The canonical restriction map $\mathrm{Sq} : \mathcal{C}_{Q,r}(X) \to \mathcal{S}^{1-\mathrm{geom}}_{Q,r}(X)$ is bijective.

Drinfeld's proof goes in three steps,

(1) every geometric Q-skeleton arises from a Q-coefficient (Lemma (4.1.2)),

(2) every 1-geometric Q-skeleton is geometric over a dense open subset $U \subset X$ (Lemma (4.1.3)),

(3) if a 1-geometric Q-skeleton coincides with a Q-coefficient over a dense open subset, then it is actually a Q-coefficient (Lemma (4.1.4)).

The proofs of these lemmas rely on the Galois description of the category of Q-coefficients.

Lemma (4.1.2) (after M. Kerz) The canonical restriction map $\mathcal{C}_{Q,r}(X) \to \mathcal{S}_{Q,r}^{\mathrm{geom}}(X)$ is bijective.

Lemma (4.1.3) (after G. Wiesend, M. Kerz-A. Schmidt, cf. in particular [KS09a, §3, §4]) For any $\mathcal{C} \in \mathcal{S}_{Q,r}^{1-\mathrm{geom}}(X)$, there exists a non-empty open subset $U \subset X$ such that $\mathcal{C}|_U \in \mathcal{S}_{Q,r}^{\mathrm{geom}}(U)$.

Lemma (4.1.4) For any non-empty open subset $U \subset X$, the following diagram (where the arrows are the canonical restriction maps) is Cartesian

$$\begin{tikzcd}
\mathcal{C}_{Q,r}(X) \arrow[r] \arrow[d] & \mathcal{S}_{Q,r}^{1-\mathrm{geom}}(X) \arrow[d] \\
\mathcal{S}_{Q,r}^{\mathrm{geom}}(U) \arrow[r] & \mathcal{S}_{Q,r}^{1-\mathrm{geom}}(U)
\end{tikzcd}$$

4.2 Proof of Lemma (4.1.2)

Let $\mathcal{C}$ be a geometric Q-skeleton of rank r over X. We need to construct a continuous morphism $\rho : \pi_1(X) \to GL(r, Q)$ such that $\det(1 - \rho(F_x)\, T) = \chi_x(\mathcal{C}, T)$ for all $x \in |X_0|$.

Let Π be as in (3.2.6)(2). Since $\pi_1(X)/\Pi$ is topologically of finite type, so the topological space

$$H := \mathrm{Hom}\left(\pi_1(X)/\Pi, \mathrm{GL}_r(Z)\right) = \varprojlim_n \mathrm{Hom}\left(\pi_1(X)/\Pi, \mathrm{GL}_r\left(Z/\lambda^n\right)\right),$$

equipped with the induced topology gien by the product of discrete topologies, it is compact.

For $x \in |X_0|$, let $H_x \subset H$ be the subset of representations $\rho : \pi_1(X)/\Pi \to \mathrm{GL}(r, Z)$ such that $\det(1 - \rho(F_x)\, T) = \chi_x(\mathcal{C}, T)$. We want to show that $\bigcap_{x \in |X|} H_x \neq \emptyset$. By compactness and since H_x is closed, it suffices to show that for any finite subset $F \subset |X|$, $\bigcap_{x \in F} H_x \neq \emptyset$. According to Theorem [Cad18, Theorem 6.1, Bertini Argument], there exists $\phi : C \to X \in \mathrm{Cu}(X)$ and a section $F \to C$ such that the induced morphism $\pi_1(C) \to \pi_1(X) \to \pi_1(X)/\Pi$ is surjective. Let $c \in |C|$ and $\rho_C : \pi_1(C) \to \mathrm{GL}(\mathcal{C}_{Cc})$ be the representation associated with $\mathcal{C}_C$. We want to show the representation $\rho_C : \pi_1(C) \to \mathrm{GL}(\mathcal{C}_{Cc})$ factors through $\rho_X : \pi_1(X)/\Pi \to \mathrm{GL}(\mathcal{C}_{Cc})$.

$$K_\Pi := \ker\left(\pi_1(C) \to \pi_1(X) \to \pi_1(X)/\Pi\right).$$

Then it suffices to show $\rho(K_\Pi)$ is trivial. Since K_Π is normal in $\pi_1(C)$ and the action of $\pi_1(C)$ on $\mathcal{C}_{Cc}$ is semi-simple, the action of K_Π on $\mathcal{C}_{Cc}$ is also semisimple by Clifford's theorem.

To show that it is trivial, it suffices to demonstrate that it is unipotent or, equivalently, that $\det(1 - \rho_C(g)\, T \mid \mathcal{C}_{Cc}) = (1 - T)^r$ for $g \in K_\Pi$. Using the notation from (3.2.6), if $C_n \to C$ is the Galois covering corresponding to the inverse image of $\pi_1(X_n)$ in $\pi_1(C)$, we have from (3.2.6)(1) that $\chi_c(\mathcal{C}, T) \equiv (1 - T)^r \left[\lambda^n\right]$ for $c \in |C_n|$. Therefore, by Čebotarev density and continuity of ρ_C,

$$\det(1 - \rho_C(g)\, T \mid \mathcal{C}_{Cc}) \equiv (1 - T)^r \left[\lambda^n\right] \text{ for } g \in \pi_1(C_n).$$

Now, according to the definition of Π in (3.2.6)(2), we have $K_\Pi = \bigcap_{n \geq 1} \pi_1(C_n)$. Therefore, the representation $\rho_C : \pi_1(C) \to \mathrm{GL}(\mathcal{C}_{Cc})$ factors through $\rho_X : \pi_1(X)/\Pi \to \mathrm{GL}(\mathcal{C}_{Cc})$. By construction, we get a $\rho_X \in H_x$.

4.3 Proof of Lemma (4.1.3)

1. We can assume that for all $C \in \mathrm{Cu}(X), \mathcal{C}_C$ is tame.

By replacing X with a dense open subset and defintion of 1-geometric, we can assume that $\mathcal{C}$ is tame by a connected étale covering as in (3.2.2). Clearly, if $X' \to X$ is an étale covering and $\mathcal{C}$ is a Q-skeleton over X, then $\mathcal{C}$ is 1-geometric (resp. geometric) if and only if $\mathcal{C}|_{X'}$ is 1-geometric (resp. geometric). Thus to prove (4.1.3), we can assume that for all $C \in \mathrm{Cu}(X), \mathcal{C}_C$ is tame.

2. Elementary fibration.

By replacing X with a non-empty open subset and k with a finite extension, we can assume [SGA4, XI, Prop. 3.3] that X is an elementary fibration, i.e., it factors as follows:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

with $\bar{f} : \overline{X} \to S$ projective, smooth, geometrically irreducible of relative dimension 1, $X \hookrightarrow \overline{X}$ an open immersion with dense image in each fiber, $\overline{X} \backslash X \to S$ finite and étale, and S smooth and geometrically irreducible over k. Let η be the generic point of S. By making a base change via an étale open subset $S' \to S$, we can further assume that $f : X \to S$ admits a section $g : S \to X$.

We want to construct a sequence of étale coverings $X_n \to X, n \geq 1$ satisfying (3.2.6). We will proceed by induction on the dimension of X.

The case where X is a curve is tautological. We therefore assume that X is of dimension $d \geq 2$.

3. Definition of $N_{n,\bar{\eta}} \subset \pi_1(X_{\bar{\eta}})$.

For each $s \in S$, choose a geometric point $\bar{s}$ above $g(s)$. Since the tame fundamental group $\pi_1^t(X_{\bar{s}})$ is topologically finitely generated [SGA1, XIII.2.12], it has only finitely many (open) subgroups of bounded index. In particular, the intersection $N_{n,\bar{s}}^t \subset \pi_1^t(X_{\bar{s}})$ of all open subgroups of index $\leq \left|\mathrm{GL}_r\left(Z/\lambda^n\right)\right|$ is still an open subgroup. Let $N_{n,\bar{\eta}} \subset \pi_1(X_{\bar{\eta}})$ be the inverse image of $N_{n,\bar{\eta}}^t \subset \pi_1^t(X_{\bar{\eta}})$ under the canonical projection $\pi_1(X_{\bar{\eta}}) \to \pi_1^t(X_{\bar{\eta}})$.

4. $N_{n,\bar{\eta}} \subset \pi_1(X_{\bar{\eta}})$ is normal.

Since $\ker(\pi_1(X_{\bar{\eta}}) \to \pi_1^t(X_{\bar{\eta}}))$ is normalized by the action of $\pi_1(\eta)$ on $\pi_1(X_\eta) \supset \pi_1(X_{\bar{\eta}})$ via the section $\pi_1(\eta) \to \pi_1(X_\eta)$ induced by $g : S \to X$, and since $N_{n,\bar{\eta}}^t \subset \pi_1^t(X_{\bar{\eta}})$ is characteristic, $N_{n,\bar{\eta}} \subset \pi_1(X_\eta)$ is a normal subgroup. Let $N_{n,\eta} := N_{n,\bar{\eta}} \rtimes_g \pi_1(\eta) \subset \pi_1(X_\eta) = \pi_1(X_{\bar{\eta}}) \rtimes_g \pi_1(\eta)$; it is an open, normal subgroup of $\pi_1(X_\eta)$.

Since S is normal, π_1 commutes with limit, so $\pi_1(X_\eta) = \lim_{V \subset S} \pi_1(X \times_S V)$, where the limit is taken over all non-empty Zariski opens $V \subset S$. By replacing S with a non-empty open, we can assume that $N_{n,\eta}$ contains the kernel of $p : \pi_1(X_\eta) \to \pi_1(X)$, and thus, by setting $N_n := p(N_{n,\eta})$, we have $N_{n,\eta} = p^{-1}(N_n)$.

Let $\widetilde{X}_n \to X$ be the Galois (connected étale) covering corresponding to $N_n \subset \pi_1(X)$. For a closed point $s \in |S|$, let $S_{(\bar{s})} := \mathrm{Spec}(\mathcal{O}_{S,\bar{s}})$ be the strict henselization and $X_{(\bar{s})} := X \times_S S_{(\bar{s})}$. We have $\pi_1^t\left(X_{(\bar{s})}\right) \simeq \pi_1^t(X_{\bar{s}})$ by [SGA1, VIII 2.10]. The theory of specialization of the tame fundamental group [SGA1, XIII] provides a commutative diagram

$$
\begin{tikzcd}
& \pi_1^t\left(X_{\bar\eta}\right) \arrow[r, two heads] \arrow[rr, bend left, "\mathrm{sp}_{\bar s}"] \arrow[dddl, "\phi_\eta"'] & \pi_1^t\left(X_{(\bar s)}\right) & \pi_1^t\left(X_{\bar s}\right) \arrow[l, "\simeq"'] \arrow[ddd, bend left] \\
& & \pi_1\left(X_{(\bar s)}\right) \arrow[u, two heads] \arrow[d] & \\
& \pi_1\left(X_{\bar\eta}\right) \arrow[uu, two heads] \arrow[r] \arrow[d, two heads] & \pi_1(X) \arrow[d] & \pi_1\left(X_{\bar s}\right) \arrow[l] \arrow[uu, two heads] \arrow[d, two heads, "\phi_s"] \\
\pi_1^t\left(X_{\bar\eta}\right)/N_{n,\bar\eta}^t \arrow[r, "\simeq"] & \pi_1\left(X_{\bar\eta}\right)/N_{n,\bar\eta} \arrow[r, "\simeq"] & \pi_1(X)/N_n & \pi_1^t\left(X_{\bar s}\right)/N_{n,\bar s}^t
\end{tikzcd}
$$

By definition of $N_{n,\bar\eta}^t, N_{n,\bar s}^t, \mathrm{sp}_{\bar s}\left(N_{n,\bar\eta}^t\right) \subset N_{n,\bar s}^t$, the commutativity of the diagram shows that $\pi_1\left(\widetilde{X}_{n,\bar s}\right) \subset \ker\left(\phi_s\right)$. In particular, $\mathcal{C}_{X_s}|_{\widetilde{X}_{n,\bar s}}$ is trivial modulo λ^n, i.e., the representation of $\pi_1\left(\widetilde{X}_{n,s}\right)$ on $\mathcal{C}_{X_s,\bar s}$ mod λ^n factors through $\pi_1\left(\widetilde{X}_{n,s}\right) \to \pi_1(s)$. But by the induction hypothesis, there exists a Galois covering $S_n \to S$ for $g^*\mathcal{C}$ as in (3.2.6) (1). Any connected component X_n of $\widetilde{X}_n \times_S S_n$ then provides a Galois covering for $\mathcal{C}$ as in (3.2.6) (1).

As it stands, the argument does not work because the open set by which we must replace S to ensure that $N_{n,\eta}$ contains the kernel of $p: \pi_1\left(X_\eta\right) \to \pi_1(X)$ depends on n. Therefore, it needs some modification:

1. We apply it as it is for $n=1$.

2. This allows us to reduce to the case where the $\mathcal{C}_C$ mod λ, $C \in \mathrm{Cu}(X)$ are trivial, and thus, in the above argument, the tame fundamental group can be replaced in the argument above by its pro ℓ-completion $\pi_1^t\left(X_{\bar s}\right) \to \pi_1^{(\ell)}\left(X_{\bar s}\right)$ for which the specialization maps $\pi_1^{(\ell)}\left(X_{\bar\eta}\right) \to \pi_1^{(\ell)}\left(X_{\bar s}\right)$ are isomorphisms, and we now use [SGA1, XIII]

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1^{(\ell)}\left(X_{\bar\eta}\right) \arrow[r] \arrow[d, "="] & \pi_1^{[\ell]}\left(X_\eta\right) \arrow[r] \arrow[d, two heads, "p"] & \pi_1(\eta) \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1^{(\ell)}\left(X_{\bar\eta}\right) \arrow[r] & \pi_1^{[\ell]}(X) \arrow[r] & \pi_1(S) \arrow[r] & 1
\end{tikzcd}
$$

where $\pi_1^{[\ell]}(-)$ is the notation for the quotient by the characteristic subgroup $\ker\left(\pi_1\left(X_{\bar\eta}\right) \to \pi_1^{(\ell)}\left(X_{\bar\eta}\right)\right)$.

3. In this setup, the group $N_{n,\eta}^{(\ell)} = N_{n,\bar\eta}^{(\ell)} \rtimes \pi_1(\eta) \subset \pi_1^{[\ell]}\left(X_\eta\right) = \pi_1^{(\ell)}\left(X_{\bar\eta}\right) \rtimes \pi_1(\eta)$ contains the kernel of $p: \pi_1^{[\ell]}\left(X_\eta\right) \to \pi_1^{[\ell]}(X)$, and therefore we can define $N_n^{[\ell]} := p\left(N_{n,\eta}^{(\ell)}\right) \subset \pi_1^{[\ell]}(X)$ and take the inverse image of $N_n^{[\ell]}$ via $\pi_1(X) \to \pi_1^{[\ell]}(X)$ without shrinking S. Furthermore, the open subgroups $\pi_1\left(X_n\right) = \pi_1\left(\widetilde{X}_n\right) \cap p^{-1}\left(\pi_1\left(S_n\right)\right)$ satisfy

$$
\pi_1(X)/\pi_1\left(X_n\right) \hookrightarrow \pi_1(X)/\pi_1\left(\widetilde{X}_n\right) \times \pi_1(S)/\pi_1\left(S_n\right) \simeq \pi_1^{(\ell)}\left(X_{\bar\eta}\right)/N_{n,\bar\eta}^{(\ell)} \times \pi_1(S)/\pi_1\left(S_n\right).
$$

Therefore $\pi_1^{(\ell)}\left(X_{\bar\eta}\right)$, and hence $\pi_1^{(\ell)}\left(X_{\bar\eta}\right)/\cap_{n\geq 1} N_{n,\bar\eta}^{(\ell)}$, is pro-$\ell$ topologically of finite type, and $\pi_1(S)/\cap_{n\geq 1}\pi_1\left(S_n\right)$ is almost pro-ℓ topologically of finite type by the induction hypothesis. Thus the $X_n \to X$ also satisfy (3.2.6) (2).

4.4 Proof of Lemma (4.1.4)

Let $\mathcal{C} \in \mathcal{S}_{Q,r}^{\mathrm{geom}}(U)$ and $\mathcal{S} \in \mathcal{S}_{Q,r}^{1-\mathrm{geom}}(X)$ such that $\mathcal{S}|_U \simeq \mathcal{C}$. According to Lemma (4.1.2), we know that $\mathcal{C} \in \mathcal{C}_{Q,r}(U)$. We need to show that $\mathcal{C}$ extends to a lisse Q-sheaf on X, still denoted by $\mathcal{C}$, and for every $x \in X \backslash U$, $\chi_x(\mathcal{C}, T) = \chi_x(\mathcal{S}, T)$. Actually the latter condition is automatic once $\mathcal{C}$ extends.

Indeed, let $x \in X \backslash U$. According to [Dri18, Theorem 2.15], there exists a smooth, geometrically connected curve C over k equipped with a morphism $\phi: C \to X$ and a $k(x)$-point c above x, such that $\phi^{-1}(U) \neq \emptyset$. In particular, $\mathcal{C}|_C$

and $\mathcal{S}|_C$ are two lisse Q-sheaves whose semi-simplifications coincide on the non-empty open subset $\phi^{-1}(U) \subset C$. By the Čebotarev theorem and since C is smooth (hence normal and we have $\pi_1(U) \twoheadrightarrow \pi_1(X)$), their semi-simplifications coincide on the entire curve C and, in particular, at c.

It remains to show that $\mathcal{C}$ extends to a lisse Q-sheaf on X. Suppose it is not. By assumption, for any $\phi : C \to X \in \mathrm{Cu}(X)$, we know that the semi-simplification of $\mathcal{C}|_{\phi^{-1}(U)}$ extends to C. Since X is smooth over k, the Zariski-Nagata purity theorem [SGA2, X, Theorem 3.4] implies that $\mathcal{C}$ must be ramified along an irreducible divisor $D \subset X \backslash U$. By replacing X with an open subset, we can assume that $D = X \backslash U$.

Lemma (4.4.1) (based on G. Wiesend, M. Kerz-A. Schmidt, c.f. in particular [KS09b, Lemma 2.4], [KS09a, Proposition 2.3]). There exists $x \in D$ and a line $\mathfrak{l}_x \subset T_x X$ (depending on $\mathcal{C}$) satisfying the following property: for every $(C, \phi) \in \mathrm{Cu}(X)$ such that $\phi^{-1}(U) \neq \emptyset$ and a point $c \in C$ above x such that $\mathrm{im}(T_c\phi) = \mathfrak{l}_x$, the sheaf $\mathcal{C}|_{\phi^{-1}(U)}$ is ramified at c.

Proof. Assume that $\mathcal{C}$ arises from a scalar extension of a lisse Z-sheaf $\mathcal{H}$, where Z is the ring of integers of a finite extension Q_λ of $\mathbb{Q}_\ell$ inside Q, and pick $n \geq 1$ such that $\mathcal{H}/\lambda^n$ is ramified along D.

If $f : U' \to U$ is the étale covering trivializing $\mathcal{H}/\lambda^n$, and $f : X' \to X$ is the normalization of X in $f : U' \to U$, it suffices to construct $x \in D$ and $\mathfrak{l}_x \subset T_x X$ such that for every $(C, \phi) \in \mathrm{Cu}(X)$ as in this lemma, the covering $f : C' := X' \times_X C \to C$ is also ramified at c.

Let G be the Galois group of $f : U' \to U$, and $I \subset G$ be the inertia group along D. By replacing X with X'/I, we can assume that $G = I$. In particular, G is solvable since I is. Therefore, by replacing $X' \to X$ with $X'/J \to X$ for a subgroup $J \subset G$, we can assume that G has prime order p. By replacing X with an open subset (whose complement has codimension greater than 2), we can assume that X' is smooth over k. Let D' be the support of the inverse image of D in X'. The hypothesis $G = I$ implies that the action of G on D' is trivial, so the covering $f : D' \to D$ is purely inseparable of degree $\mathfrak{f} \mid |G| = p$.

It suffices to construct $\phi : C \to X$ such that C' is smooth at $f^{-1}(c)$ (the covering $C' \to C$ will then have only one point above c and thus be ramified). To achieve this, it suffices for $\mathfrak{l}_x$ to be transverse to $H_x := \mathfrak{m}(T_{f^{-1}(x)} f)$.

1. If $\mathfrak{f} = 1$, it is an isomorphism. We can then take any $x \in D$ and $\mathfrak{l}_x \not\subset T_x D$.

2. If $\mathfrak{f} = p$, by replacing X with an open subset (whose complement has codimension greater than 3), we can assume that D and D' are smooth and $H_x \subset T_x D$ has codimension 1 for all $x \in D$. Then it suffices to take $x \in D$ and $\mathfrak{l}_x \subset T_x D$, $\mathfrak{l}_x \not\subset H_x$. □

Let's go back to the proof of Lemma (4.1.4). The Bertini-Poonen theorem [Poo04] ensures the existence of (C, ϕ) with this property. Using Lemma (4.4.1), since $\mathcal{C}$ is lisse and semi-simple, it is a direct sum of $\overline{\mathbb{Q}}_\ell$-pure sheaves (this follows from the combination of Theorem (A.1.8) and companion conjecture for curve). The same holds for $\mathcal{C}|_{\phi^{-1}(U)}$, which ensures that $\mathcal{C}|_{\phi^{-1}(U)_{\bar{k}}}$ is semisimple(Proposition (A.1.12)), so it is unramified. Since being ramified is stable under base change, this shows that $\mathcal{C}|_{\phi^{-1}(U)}$ extends to C unramified at c, this gives a contradiction.

5 Moduli Space and Finiteness ([EK12, 6])

Let X be smooth separated scheme over $\mathbb{F}_q$ of finite type over the finite field. Assume that there is a connected normal projective compactification $X \subset \overline{X}$ such that $\overline{X} \backslash X$ is the support of an effective Cartier divisor on $\overline{X}$.

(5.0.1) In (A.1.13), we define $\mathcal{P}_r := \mathbb{G}_m \times \mathbb{A}^{r-1}$. Let $\mathcal{L}_r(X)$ be the product $\prod_{|X|} \mathcal{P}_r$ with one copy of $\mathcal{P}_r$ for every closed point of X. It is an affine scheme over $\mathbb{Q}$, which if $\dim(X) \geq 1$ is not of finite type over $\mathbb{Q}$. We have an injective map defined in (3.1.3)

$$\begin{aligned}\kappa : \mathcal{S}_{\overline{\mathbb{Q}}_\ell,r}(X) &\to \mathcal{L}_r(X)\left(\overline{\mathbb{Q}}_\ell\right),\\ \left(\mathcal{C}_C\right)_{C\in \mathrm{Cu}(X)} &\mapsto \prod_{|X|} \chi_x(\mathcal{C}_{C_x}, T).\end{aligned} \tag{5.1}$$

The existence of the moduli space of ℓ-adic sheaves on X is shown in the following theorem of Deligne.

Theorem (5.0.2) For any effective Cartier divisor $D \in \mathrm{Div}^+(\overline{X})$ with support in $\overline{X}\backslash X$ there is a unique reduced closed subscheme $L_r(X, D)$ of $\mathcal{L}_r(X)$ which is of finite type over $\mathbb{Q}$ and such that

$$L_r(X,D)\left(\overline{\mathbb{Q}}_\ell\right) = \kappa\left(\mathcal{S}_{\overline{\mathbb{Q}}_\ell,r}(X,D)\right).$$

To construct the (coarse) moduli space, we need Deligne's finiteness theorem [EK12, Theorem 2.1]:

Theorem (5.0.3) Let X be connected and $D \in \mathrm{Div}^+(\overline{X})$ be an effective Cartier divisor with support in $\overline{X}\backslash X$. The set of irreducible sheaves $V \in \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)$ is finite up to twist by elements of $\mathcal{C}_{\overline{\mathbb{Q}}_\ell,1}(\mathbb{F}_q)$.

We give a sketch of the construction of the moduli space.

5.1 Moduli over Curves

In this section we assume that X is a curve.

Step 1. For any $V_1 \oplus \cdots \oplus V_n \in \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)$ and the map

$$\begin{aligned}\left(\mathcal{C}_{\overline{\mathbb{Q}}_\ell,1}(\mathbb{F}_q)\right)^n &\to \mathcal{L}_r(X)\left(\overline{\mathbb{Q}}_\ell\right),\\ (\chi_1,\ldots,\chi_n) &\mapsto \kappa\left(\chi_1\cdot V_1 \oplus \cdots \oplus \chi_n \cdot V_n\right).\end{aligned} \tag{5.2}$$

It can be shown that this map is induced by a finite morphism between of affine schemes [EK12, Lemma 6.2]. Then by [EK12, Proposition A.3], we obtain the existence of a unique reduced closed subscheme L_i of $\mathcal{L}_r(X)\otimes\overline{\mathbb{Q}}_\ell$ of finite type over $\overline{\mathbb{Q}}_\ell$ such that $L_i\left(\overline{\mathbb{Q}}_\ell\right)$ is the image of the map (5.2).

Step 2.

By Theorem (5.0.3), there are only finitely many direct sums

$$V_1 \oplus \cdots \oplus V_n \in \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D) \tag{5.3}$$

with V_i irreducible up to twists $\chi_i \mapsto \chi_i \cdot V_i$. Let

$$L_r(X,D)_{\overline{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}_\ell$$

be the reduced scheme, which is the union of the finitely many closed subschemes $L_i \hookrightarrow \mathcal{L}_r(X)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}_\ell$ corresponding to representatives of the finitely many twisting classes of direct sums (5.3). Clearly $L_r(X,D)_{\overline{\mathbb{Q}}_\ell}\left(\overline{\mathbb{Q}}_\ell\right) = \kappa\left(\mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)\right)$ and $L_r(X,D)_{\overline{\mathbb{Q}}_\ell}$ is of finite type over $\overline{\mathbb{Q}}_\ell$.

Step 3.

The automorphism group $\mathrm{Aut}\left(\overline{\mathbb{Q}}_\ell/\mathbb{Q}\right)$ acting on $\mathcal{L}_r(X)$ stabilizes $\kappa\left(\mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)\right)$ by [EK12, Corollary 4.9] (A corollary of companion theorem from [Laf02]). Using the descent Proposition [EK12, A.2] the scheme $L_r(X,D)_{\overline{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}_\ell$ over $\overline{\mathbb{Q}}_\ell$ descends to a closed subscheme $L_r(X,D)\hookrightarrow \mathcal{L}_r(X)$.

5.2 Higher Dimension

In general case, it is easy to construct a closed subscheme $L_r(X,D) \hookrightarrow \mathcal{L}_r(X)$ that

$$L_r(X,D)\left(\overline{\mathbb{Q}}_\ell\right) = \kappa\left(\mathcal{S}_{\overline{\mathbb{Q}}_\ell,r}(X,D)\right)$$

relying on construction for curves. However from this construction it is not clear that $L_r(X,D)$ is of finite type over $\mathbb{Q}$. Actually, we define the reduced closed subscheme $L_r(X,D) \hookrightarrow \mathcal{L}_r(X)$ by the Cartesian square (in the category of reduced schemes)

$$\begin{tikzcd}
L_r(X,D) \arrow[r] \arrow[d] & \mathcal{L}_r(X) \arrow[d] \\
\prod_{C\in \mathrm{Cu}(X)} L_r\left(C,\bar{\phi}^*(D)\right) \arrow[r] & \prod_{C\in \mathrm{Cu}(X)} \mathcal{L}_r(C)
\end{tikzcd}$$

From the definition of $\mathcal{S}_r(X,D)$ and construction for curve we get

$$L_r(X,D)\left(\overline{\mathbb{Q}}_\ell\right) = \kappa\left(\mathcal{S}_{\overline{\mathbb{Q}}_\ell,r}(X,D)\right).$$

In addition, as $\mathcal{L}_r(X) \to \prod_{C\in \mathrm{Cu}(X)} \mathcal{L}_r(C)$ is a closed immersion, so is $\mathcal{L}_r(X,D) \to \prod_{C\in \mathrm{Cu}(X)} L_r\left(C,\bar{\phi}^*(D)\right)$ by base change. To show that it is of finite type, see [EK12, 6.3].

6 Application of Companion Theorem: Tannakian Čebotarev Density Theorem (A. Cadoret)

Let X_0 be a smooth variety, separated and of finite type over $\mathbb{F}_q$. Let $\mathcal{C}_0$ be a semisimple Q-coefficient on X_0, where we fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

(6.0.1) Given a geometric point x over x_0, recall that we define $G(\mathcal{C}_0) \subset GL(\mathcal{C}_x)$ as the Zariski closure of the images of $\pi_1(X_0)$ acting on $\mathcal{C}_x$. It's easy to see that the image is independent of the choice of x. We define $\Phi_{x_0}^{\mathcal{C}_0}$ as the $G(\mathcal{C}_0)$-conjugacy class of the image of the geometric Frobenius F_{x_0} by the map $\pi_1(X_0) \to G(\mathcal{C}_0)$. For a subset $S \subset |X_0|$ of closed points, write $\Phi_S^{\mathcal{C}_0} := \cup_{x_0\in S}\Phi_{x_0}^{\mathcal{C}_0}$ and for every $G(\mathcal{C}_0)$-invariant subset $\Delta \subset G(\mathcal{C}_0)$, write $S_\Delta^{\mathcal{C}_0} := \left\{x_0 \in S \mid \Phi_{x_0}^{\mathcal{C}_0} \subset \Delta\right\}$. In the following, we will also omit the superscript $(-)^{\mathcal{C}_0}$ from the notation.

For $S \subseteq T$ two sets of positive integers, with T infinite, the upper natural density of S in T are defined as

$$\limsup_{N\to\infty} \frac{\#\{n \in S : n \leq N\}}{\#\{n \in T : n \leq N\}}. \tag{6.1}$$

Theorem (6.0.2) Assume S has upper Dirichlet density $\delta^u(S) > 0$. Then the Zariski-closure of Φ_S contains at least one connected component of $G(\mathcal{C}_0)$.

Firstly we prove this theorem for étale $\overline{\mathbb{Q}}_\ell$-coefficient, then use companion theorem to reduce general case to the étale case.

6.1 Étale $\overline{\mathbb{Q}}_\ell$-Coefficient

For étale $\overline{\mathbb{Q}}_\ell$-coefficient, Theorem (6.0.2) simply follows just from classical Čebotarev density theorem.

Write $G_0 := G(\mathcal{C}_0)$, which we identify with the Zariski-closure of the image Π_0 of the continuous representation V of $\pi_1(X_0)$ corresponding to $\mathcal{C}_0$. For every closed point $x_0 \in |X_0|$, let $\widetilde{\Phi}_{x_0} \subset \Pi_0$ denote the Π_0-conjugacy class of the image φ_{x_0} of a geometric Frobenius attached to x_0 so that the G_0-conjugacy class Φ_{x_0} defined in (6.0.1) is the Zariski-closure of $\widetilde{\Phi}_{x_0}$.

For every closed subset $C \subset G_0$ which is a union of conjugacy classes, the subset S_C is defined as

$$S_C = \left\{ x_0 \in |X_0| \mid \widetilde{\Phi}_{x_0} \subset \Pi_0 \cap C \right\}. \tag{6.2}$$

Without loss of generality one may assume $\mathcal{C}_0$ is defined over a finite extension Q_ℓ of $\mathbb{Q}_\ell$. Let Z_ℓ denote the ring of integers of Q_ℓ. Fix a Π_0-stable Z_ℓ-lattice $\Lambda \subset V$, set $G_0(Z_\ell) := G_0(Q_\ell) \cap GL(\Lambda)$ and let $\mu : \mathcal{B}(G_0(Z_\ell)) \to [0, |\pi_0(G_0)|]$ denote the Haar measure on $G_0(Z_\ell)$ normalized so that $\mu(G_0^\circ(Z_\ell)) = 1$, where $\mathcal{B}(G_0(Z_\ell))$ denotes the Borel algebra on $G_0(Z_\ell)$. Assume $C := \overline{\Phi_S}^{zar}$ does not contain any connected component of G_0. Since $G_0(Z_\ell)$ is Zariski-dense in $G_0, \mu(C(Z_\ell)) = 0$ [Ser16, Proposition 5.12]. On the other hand, since $C(Z_\ell) \subset G_0(Z_\ell)$ is analytically closed, we have $0 < \delta^u(S) \leq \delta^u\left(|X_0|_{C(Z_\ell)}\right) \leq \mu(C(Z_\ell)) = 0$, where the last inequality is [Ser16, Theorem 6.8] (using the description (6.2) of $|X_0|_{C(Z_\ell)}\big)$, whence a contradiction.

6.2 Semisimple $\overline{\mathbb{Q}}_\ell$-Coefficient

We aim to use the companion theorem to reduce the general $\overline{\mathbb{Q}}_\ell$-coefficient to the case of étale $\overline{\mathbb{Q}}_\ell$-coefficient. By [Cad18, §5], A. Cadoret reformulates Theorem (6.0.2) in terms of the characteristic polynomial map attached to $\mathcal{C}_0$, so for the general semisimple $\overline{\mathbb{Q}}_\ell$-coefficient, it suffices to prove:

Proposition (6.2.1) Assume that $\mathcal{C}_0$ is a semisimple $\overline{\mathbb{Q}}_\ell$-coefficient. Then for every prime $p \neq \ell$ large enough, there exists an isomorphism $\iota' : \overline{\mathbb{Q}}_{\ell'} \to \mathbb{C}$ and a (necessarily unique) semisimple étale $\overline{\mathbb{Q}}_{\ell'}$-coefficient $\mathcal{C}_0'$ which is the companion (with respect to ι, ι') of $\mathcal{C}_0$.

Proof. From [Del80, 1.3.8], one can write $\mathcal{C}_0 = \oplus_{i\in I}\mathcal{I}_{i,0}^{(\alpha_i)}$ with $\mathcal{I}_{i,0}$ irreducible with finite determinant and $\alpha_i \in \overline{\mathbb{Q}}_\ell^\times, i \in I$, and $\mathcal{I}_{i,0}^{(\alpha_i)}$ is the twist, see (A.1.5). By (1.6.1) (3), for every $\ell' \neq p$ and isomorphism $\iota' : \overline{\mathbb{Q}}_{\ell'} \to \mathbb{C}$ and for every $i \in I$, there exists an étale $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{I}_{i,0,\ell'}$ compatible with $\mathcal{I}_{i,0}$. What's more, $\mathcal{I}_{i,0,\ell'}$ is irreducible hence, by construction, $\mathcal{C}_0' := \oplus_{i\in I}\mathcal{I}_{i,0,\ell'}^{\left(\iota'^{-1}\iota(\alpha_i)\right)}$ is a semisimple $\overline{\mathbb{Q}}_{\ell'}$-coefficient on X_0 compatible with $\mathcal{C}_0$. From lemma (6.2.2) below, for $p \neq \ell$ large enough, one can furthermore choose $\iota' : \overline{\mathbb{Q}}_{\ell'} \simeq \mathbb{C}$ in such a way that the $\iota'^{-1}\iota(\alpha_i)$ are ℓ-adic units that is $\mathcal{C}_0'$ is an étale $\overline{\mathbb{Q}}_{\ell'}$-coefficient. □

Lemma (6.2.2) Let $0 \neq \alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Then for every prime ℓ' large enough, there exists a field isomorphism $\iota' : \overline{\mathbb{Q}}_{\ell'} \simeq \mathbb{C}$ such that $\iota'^{-1}(\alpha_1), \ldots, \iota'^{-1}(\alpha_m)$ are ℓ'-adic units.

Proof. By the Noether Normalization lemma there exists elements $t_1, \ldots, t_r \in \mathbb{Q}\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]$, algebraically independent over $\mathbb{Q}$ and such that the extension $\mathbb{Q}[t_1, \ldots, t_r] \subset \mathbb{Q}\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]$ is finite. For some integer $N \geq 1$, the extension $\mathbb{Q}[t_1, \ldots, t_r] \hookrightarrow \mathbb{Q}\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]$ extends to a finite extension $\mathbb{Z}[1/N][t_1, \ldots, t_r] \hookrightarrow \mathbb{Z}[1/N]\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]$.

Fix a prime $\ell' \nmid N$. Since $\mathbb{Z}_\ell'$ is uncountable, one can find $t_{1,\ell'}, \ldots, t_{r,\ell'} \in \mathbb{Z}_{\ell'}$ algebraically independent over $\mathbb{Q}$, whence an embedding $\mathbb{Z}[1/N][t_1, \ldots, t_r] \hookrightarrow \mathbb{Z}_{\ell'}$. Localizing at the zero-ideal, one obtains a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_{\ell'} \arrow[d, hook] & \mathbb{Z}[1/N][t_1, \ldots, t_r] \arrow[l, hook'] \arrow[r, hook, "\text{finite}"] \arrow[d, hook] & \mathbb{Z}[1/N]\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right] \arrow[d, hook] \\
\mathbb{Q}_{\ell'} & \mathbb{Q}(t_1, \ldots, t_r) \arrow[l, hook'] \arrow[r, hook, "\text{finite}"] & \mathbb{Q}(\alpha_1, \ldots, \alpha_m)
\end{tikzcd}$$

hence, taking a connected component $E_{\ell'}$ of $\mathbb{Q}(\alpha_1, \ldots, \alpha_m) \otimes_{\mathbb{Q}(t_1,\ldots,t_r)} \mathbb{Q}_{\ell'}$, a commutative diagram of fields

$$\begin{tikzcd}
\mathbb{Q}(t_1, \ldots, t_r) \arrow[r, hook, "\text{finite}"] \arrow[d, hook] & \mathbb{Q}(\alpha_1, \ldots, \alpha_m) \arrow[d, hook] \\
\mathbb{Q}_{\ell'} \arrow[r, hook, "\text{finite}"] & E_{\ell'}
\end{tikzcd}$$

Let $O_{\ell'}$ denote the ring of integers of $E_{\ell'}$. Since $\mathbb{Z}[1/N][t_1, \ldots, t_r] \subset \mathbb{Z}[1/N]\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]$ is finite hence proper, the valuative criterion of properness yields a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[1/N][t_1, \ldots, t_r] \arrow[r, hook, "\text{finite}"] \arrow[d, hook] & \mathbb{Z}[1/N]\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]) \arrow[d, hook] \arrow[dl, hook, dotted, "\exists!"] \\
O_{\ell'} \arrow[r, hook] & E_{\ell'}
\end{tikzcd}$$

where the diagonal dotted arrow is automatically injective. Eventually, using that $\mathbb{C}$ and $\overline{\mathbb{Q}}_{\ell'}$ have the same transcendence degree, the above diagram extends as

$$\begin{tikzcd}
\mathbb{Z}[1/N][t_1, \ldots, t_r] \arrow[r, hook, "\text{finite}"] \arrow[d, hook] & \mathbb{Z}[1/N]\left[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}\right]) \arrow[r, hook] \arrow[d, hook] \arrow[dl, hook, dotted, "\exists!"] & \overline{\mathbb{Q}}_\ell \arrow[r, "\simeq"] & \mathbb{C} \\
O_{\ell'} \arrow[r, hook] & E_{\ell'} \arrow[r] & \overline{\mathbb{Q}}_{\ell'} \arrow[ur, dotted, "\simeq"'] &
\end{tikzcd}$$

where the right up right dotted arrow is an isomorphism. $\square$

(6.2.3) Moreover, A. Cadoret proves Theorem (6.0.2) for any $\overline{\mathbb{Q}}_\ell$-coefficient without any assumption of semisimplicity [Cad18, §7] using weight theory. Moreover, Theorem (6.0.2) can be deduced easily is a straightforward consequence of the conjectural formalism of pure motives .

7 Application of the Companion Theorem: Weakly Motivic $\overline{\mathbb{Q}}_\ell$-Sheaves

7.1 Definition of Weakly Motivic $\overline{\mathbb{Q}}_\ell$-Sheaves

Let X be a scheme of finite type over $\mathbb{F}_p$. The set of its closed points will be denoted by $|X|$. Let ℓ be a prime different from p and let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Let $\mathrm{Sh}\left(X, \overline{\mathbb{Q}}_\ell\right)$ be the abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves on X and $\mathscr{D}(X, \mathbb{Q}_\ell) = D_c^b\left(X, \overline{\mathbb{Q}}_\ell\right)$ the bounded ℓ-adic derived category [Del80, §1.2-1.3].

(7.1.1) Let us consider a map

$$\Gamma : |X| \to \left\{ \text{subsets of } \overline{\mathbb{Q}}^\times \right\}.$$

Once we choose a prime $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}}_\ell \supset \mathbb{Q}_\ell$, and an embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we define a full subcategory $\mathrm{Sh}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right) \subset \mathrm{Sh}\left(X, \overline{\mathbb{Q}}_\ell\right)$: a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ is in $\mathrm{Sh}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell\right)$ if for every closed point $x \in X$, all eigenvalues of the geometric Frobenius $F_x : \mathcal{F}_x \to \mathcal{F}_x$ are in $i(\Gamma_x)$.

Let $\mathscr{D}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right) \subset \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right)$ be the full subcategory of complexes whose cohomology sheaves are in $\mathrm{Sh}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right)$. Let $K_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right)$ denote the Grothendieck group of $\mathscr{D}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right)$.

(7.1.2) For any field E, set

$$A(E) := \left\{ f \in E(t)^\times \mid f(0) = 1 \right\}.$$

A sheaf $\mathcal{F} \in \mathrm{Sh}_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right)$ defines a map

$$f_{\mathcal{F}} : |X| \to A(i(\overline{\mathbb{Q}})) = A(\overline{\mathbb{Q}}), \quad x \mapsto \det\left(1 - F_x t, \mathcal{F}\right)$$

Since for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $f_{\mathcal{F}} = f_{\mathcal{F}'} f_{\mathcal{F}/\mathcal{F}'}$, we get a homomorphism $K_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right) \to A(\overline{\mathbb{Q}})^{|X_0|}$, where $A(\overline{\mathbb{Q}})^{|X_0|}$ is the group of all maps $|X_0| \to A(\overline{\mathbb{Q}})$. This map is injective [Dri18, Lemma 1.5] by reducing to the normal case then using Čebotarev density theorem.

(7.1.3) By Drinfeld's Theorem (1.5.1) and conjecture (1.3.3) (a) (proved in [Del12]), if for each $x \in |X|$ all elements of Γ_x are units outside of p, the subgroup $K_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right) \subset A(\overline{\mathbb{Q}})^{|X|}$ does not depend on the choice of $\ell, \overline{\mathbb{Q}}_\ell$, and $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. Thus we can write simply $K_\Gamma(X, \overline{\mathbb{Q}})$ instead of $K_\Gamma\left(X, \overline{\mathbb{Q}}_\ell, i\right)$.

Definition (7.1.4) For $x \in |X|$, let $\Gamma_x^{\mathrm{mix}} \subset \overline{\mathbb{Q}}^\times$ be the set of numbers $\alpha \in \overline{\mathbb{Q}}^\times$ with the following property: there exists $n \in \mathbb{Z}$ such that all complex absolute values of α equal $q_x^{n/2}$, where q_x is the order of the residue field of x. Let Γ_x^{mot} be the set of those numbers from Γ_x^{mix} that are units outside of p.

Since Γ_x^{mix} is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the categories $\mathrm{Sh}_{\Gamma^{\mathrm{mot}}}\left(X, \overline{\mathbb{Q}}_\ell, i\right)$ and $\mathscr{D}_{\Gamma^{\mathrm{mot}}}\left(X, \overline{\mathbb{Q}}_\ell, i\right)$ do not depend on the choice of $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. We denote them by $\mathrm{Sh}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ and $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$. Similar we define categories $\mathrm{Sh}_{\mathrm{mix}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ and $\mathscr{D}_{\mathrm{mix}}\left(X, \overline{\mathbb{Q}}_\ell\right)$. The following theorem [Laf02, VII.8] tells us that $\mathrm{Sh}_{\mathrm{mix}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ coincides with the mixed ℓ-adic sheaf defined in [Del80, definition 1.2.2]. More precisely, by dévissage we reduce to the case where $\mathcal{F}$ is simple lisse sheaf, after some twist, we then can apply [Laf02, VII.7 (i)].

Definition (7.1.5) Objects of $\mathrm{Sh}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ (resp. $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$) are called weakly motivic $\overline{\mathbb{Q}}_\ell$-sheaves (resp. weakly motivic $\overline{\mathbb{Q}}_\ell$-complexes).

As (7.1.3) is applicable to Γ_x^{mot}, we have a well defined group

$$K_{\mathrm{mot}}(X, \overline{\mathbb{Q}}) := K_{\Gamma^{\mathrm{mot}}}(X, \overline{\mathbb{Q}}).$$

7.2 Grothendieck's Yoga in $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$

Lemma (7.2.1) Let $f : X \to Y$ be a morphism between schemes of finite type over $\mathbb{F}_p$. Suppose that a $\overline{\mathbb{Q}}_\ell$-sheaf M on X has the following property: the eigenvalues of the geometric Frobenius acting on each stalk of M are algebraic numbers which are units outside of p. Then this property holds for the sheaves $R^i f_! M$ and $R^i f_* M$.

Proof. The statement about $R^i f_! M$ follows from [SGA7-II, XXI. Theorem 5.2.2]. In [SGA7-II, VII. 5.0] Deligne defined "T-integer" for some set of primes T, in our case take $T = \{p\}$ and use the fact that $\Gamma_x^{\mathrm{mot}} = \Gamma_x^{\mathrm{mix}} \cap R(q)^\times$, where $R(q)$ is the integral closure of $\mathbb{Z}[1/q]$ in $\overline{\mathbb{Q}}$, so that by checking fibers we can apply [SGA7-II, XXI. Theorem 5.2.2]. (ii) follows from the proof of [SGA7-II, Theorem 5.6] and de Jong's result on alterations [Jon96, Theorem 4.1]. □

Theorem (7.2.2) Let $f : X \to Y$ be a morphism between schemes of finite type over $\mathbb{F}_p$. Then

(i) the functor $f_! : \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right) \to \mathscr{D}\left(Y, \overline{\mathbb{Q}}_\ell\right)$ maps $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ to $\mathscr{D}_{\mathrm{mot}}\left(Y, \overline{\mathbb{Q}}_\ell\right)$;

(ii) the functor $f_* : \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right) \to \mathscr{D}\left(Y, \overline{\mathbb{Q}}_\ell\right)$ maps $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ to $\mathscr{D}_{\mathrm{mot}}\left(Y, \overline{\mathbb{Q}}_\ell\right)$;

(iii) the functors f^* and $f^!$ map $\mathscr{D}_{\mathrm{mot}}\left(Y,\overline{\mathbb{Q}}_\ell\right)$ to $\mathscr{D}_{\mathrm{mot}}\left(X,\overline{\mathbb{Q}}_\ell\right)$.

Proof. (i) From [Del80, Theorem 3.3.1] we know that $f_!$ maps $\mathscr{D}_{\mathrm{mix}}\left(X,\overline{\mathbb{Q}}_\ell\right)$ to $\mathscr{D}_{\mathrm{mix}}\left(Y,\overline{\mathbb{Q}}_\ell\right)$, and (i) follows from (7.2.1) and definition.

(ii) Similar to (i), using [Del80, Theorem 6.1.2]: f_* maps $\mathscr{D}_{\mathrm{mix}}\left(X,\overline{\mathbb{Q}}_\ell\right)$ to $\mathscr{D}_{\mathrm{mix}}\left(Y,\overline{\mathbb{Q}}_\ell\right)$.

(iii) For f^* the statement is obvious. For $f^!$ it follows from (ii), we follow the proof in [Del77, Théorèmes de finitude en cohomologie ℓ-adique, Corollary 1.5]. Precisely, the problem is local, which allows us to assume that f factors as $X \xrightarrow{i} Z \xrightarrow{g} Y$ (with i a closed embedding and g smooth, purely of relative dimension n). The Poincaré duality $g^! K = K(n)[2n]$ and the transitivity $f^! = i^! g^!$ reduce the proof to the case of i. For any $L \in \mathscr{D}\left(Z,\mathbb{Q}_\ell\right)$, if j is the inclusion of $U = Z\backslash X$ into Z, then $i_* i^! L$ is the mapping cylinder of $K \to j_* j^* K$, and (iii) for i follows from (ii). ☐

Theorem (7.2.3) For any scheme X of finite type over $\mathbb{F}_p$, the full subcategory $\mathscr{D}_{\mathrm{mot}}\left(X,\overline{\mathbb{Q}}_\ell\right) \subset \mathscr{D}\left(X,\overline{\mathbb{Q}}_\ell\right)$ is stable with respect to the functor $\otimes$, the Verdier duality functor $\mathbb{D}$, and the internal $\mathscr{H}om$ functor.

Proof. The statement for $\otimes$ is obvious. The other two statements follow from Theorem (7.2.2). ☐

Definition (7.2.4) Two invertible $\overline{\mathbb{Q}}_\ell$-sheaves A and A' on $\operatorname{Spec}\mathbb{F}_p$ are equivalent if $A'A^{-1}$ is weakly motivic. Let S denote the set of equivalence classes.

The following theorem is an analogue of [Del80, Theorem 3.4.1(i)].

Theorem (7.2.5) Let $\mathscr{D}\left(X,\overline{\mathbb{Q}}_\ell\right)$ be the essential image of $\mathscr{D}_{\mathrm{mot}}\left(X,\overline{\mathbb{Q}}_\ell\right)$ under the functor of tensor multiplication by $\pi^* A$ (clearly $\mathscr{D}_A\left(X,\overline{\mathbb{Q}}_\ell\right)$ depends only on the class of A in S). Then

$$\mathscr{D}\left(X,\overline{\mathbb{Q}}_\ell\right) = \bigoplus_{A\in S} \mathscr{D}_A\left(X,\overline{\mathbb{Q}}_\ell\right). \tag{7.1}$$

Proof. The proof proceeds in two steps:

(i) The triangulated category $\mathscr{D}\left(X,\mathbb{Q}_\ell\right)$ is generated by the subcategories $\mathscr{D}_A\left(X,\overline{\mathbb{Q}}_\ell\right)$.

(ii) The subcategories $\mathscr{D}_A\left(X,\overline{\mathbb{Q}}_\ell\right)$ are orthogonal to each other.

(i) By dévissage we know that the triangulated category $\mathscr{D}\left(X,\overline{\mathbb{Q}}_\ell\right)$ is generated by objects of the from $i_!\mathcal{C}$, where $i\colon Y \hookrightarrow X$ is a locally closed embedding with Y normal (we take a normal open subscheme generically and for the complement we use dimension induction) connected and $\mathcal{C}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf on Y. So it remains to show that for any such Y and $\mathcal{C}$ there exists an invertible $\overline{\mathbb{Q}}_\ell$-sheaf A on $\operatorname{Spec}\mathbb{F}_p$ such that $\mathcal{C}\otimes\pi^*A^{-1}$ is weakly motivic. By [Del80, §1.3.6], there exists A such that the determinant of $\mathcal{C}\otimes\pi^*A^{-1}$ has finite order. Since $\mathcal{C}\otimes\pi^*A^{-1}$ is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant has finite order, it is weakly motivic (and pure) by a result of Lafforgue [Laf02, Proposition VII.7(ii)].

(ii) By the definition of orthogonality, we have to prove that if $M_1, M_2 \in \mathscr{D}_{\mathrm{mot}}\left(X,\overline{\mathbb{Q}}_\ell\right)$, A is an invertible $\overline{\mathbb{Q}}_\ell$-sheaf on $\operatorname{Spec}\mathbb{F}_p$ and $\operatorname{Ext}^i\left(M_1\otimes\pi^*A, M_2\right) \neq 0$ for some i, then $M_1\otimes\pi^*A, M_2$ are in the same class, i.e. A is weakly motivic. By the adjoint property, we have

$$\operatorname{Ext}^i\left(M_1\otimes\pi^*A, M_2\right) = \operatorname{Ext}^i\left(A, \pi_*\mathscr{H}om\left(M_1,M_2\right)\right), \tag{7.2}$$

hence $\pi_*\mathscr{H}om\left(M_1,M_2\right)$ is weakly motivic by Theorems (7.2.3) and (7.2.2)(ii). So if the LHS of (7.2) is nonzero, then A has to be weakly motivic otherwise RHS would be 0 by the following lemma. ☐

Lemma (7.2.6) $\mathrm{Ext}^i(M_1, M_2) = 0$ if M_1 belongs to $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ and M_2 does not.

Proof. By shifting it suffices to prove $\mathrm{Hom}(M_1, M_2) = 0$. We have

$$\mathrm{Hom}(M_1, M_2) \simeq H^0(\mathrm{Spec}(\mathbb{F}_q), \pi_* \mathscr{H}om(M_1, M_2)) = 0,$$

since $\pi_* \mathscr{H}om(M_1, M_2)$ is not in $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ by (7.2.1) and (7.2.3), so the eigenvalue of the geometric Frobenius F on $H^0(\mathrm{Spec}(\mathbb{F}_q), \pi_* \mathscr{H}om(M_1, M_2))$ are not in Γ_x^{mix}, but it should be Frobenius invariant. ☐

Corollary (7.2.7) One has

$$\mathrm{Sh}\left(X, \overline{\mathbb{Q}}_\ell\right) = \bigoplus_{A \in S} \mathrm{Sh}_A\left(X, \overline{\mathbb{Q}}_\ell\right), \quad \mathrm{Sh}_A\left(X, \overline{\mathbb{Q}}_\ell\right) := \mathrm{Sh}\left(X, \overline{\mathbb{Q}}_\ell\right) \cap \mathscr{D}_A\left(X, \overline{\mathbb{Q}}_\ell\right), \tag{7.3}$$

$$\mathrm{Perv}\left(X, \overline{\mathbb{Q}}_\ell\right) = \bigoplus_{A \in S} \mathrm{Perv}_A\left(X, \overline{\mathbb{Q}}_\ell\right), \quad \mathrm{Perv}_A\left(X, \overline{\mathbb{Q}}_\ell\right) := \mathrm{Perv}\left(X, \overline{\mathbb{Q}}_\ell\right) \cap \mathscr{D}_A\left(X, \overline{\mathbb{Q}}_\ell\right), \tag{7.4}$$

where S is as in Definition (7.2.4).

Proof. As $\mathrm{Sh}\left(X, \overline{\mathbb{Q}}_\ell\right) \subset \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right)$ and $\mathrm{Perv}\left(X, \overline{\mathbb{Q}}_\ell\right) \subset \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right)$ are closed under direct sums and direct summands, then it follows from Theorem (7.2.5). ☐

(7.2.8) The category $\mathrm{Perv}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right) := \mathrm{Perv}\left(X, \overline{\mathbb{Q}}_\ell\right) \cap \mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ is one of the direct summands in the decomposition (7.4) (it corresponds to the trivial A). Similarly, $\mathrm{Sh}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ is one of the summands in (7.3) and $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right)$ is one of the summands in (7.1).

Proposition (7.2.9) (i) The full subcategory $\mathscr{D}_{\mathrm{mot}}\left(X, \overline{\mathbb{Q}}_\ell\right) \subset \mathscr{D}\left(X, \overline{\mathbb{Q}}_\ell\right)$ is stable with respect to the perverse truncation functors $\tau_{\leq i}$ and $\tau_{\geq i}$.

(ii) A perverse $\overline{\mathbb{Q}}_\ell$-sheaf is weakly motivic if and only if each of its irreducible subquotients is.

Proof. (i) and (ii) are from the direct sum decomposition in (7.1), (7.4). On the other way, (i) follows from the corresponding statements for mixed sheaves in [BBD82, 5.1.6]. Precisely, "weakly motivic" is tested on the cohomology sheaf and we just need the natural truncation to construct the perverse truncation functors.

(ii) follows from the corresponding statements for mixed sheaves in [BBD82, 5.1.7]. Precisely, the sufficiency is clear, and for the necessity, use [BBD82, 5.1.3] (hypothesis of [BBD82, 5.1.3] is satisfied by (7.2.3) and (i)). ☐

A Weil II and Weights Theory

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ and $\mathbb{F} = \overline{\mathbb{F}}_q$; let ℓ denote an arbitrary prime different from p; let X_0 denote a normal, geometrically connected scheme of finite type over $\mathbb{F}_q$ and $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$. Let $|X_0|$ denote the set of closed points of X_0. For $x \in |X_0|$, let $\kappa(x)$ denote the residue field of x and let $q(x)$ denote the cardinality of $\kappa(x)$.

A.1 Q-Coefficient

(A.1.1) Let R be a local Noetherian ring of residual characteristic ℓ, and $\mathfrak{m}$ its maximal ideal. We assume that R is complete with respect to the $\mathfrak{m}$-adic topology. The category of constructible R-sheaves is the 2-limit of the categories [SGA4, VI (6.10)] of sheaves of R/I-modules, where I is an open ideal of R.

Let E be a finite extension of $\mathbb{Q}_\ell$ and R be the integral closure of $\mathbb{Z}_\ell$ in E. The category of constructible E-sheaves is the quotient of the category of constructible R-sheaves by the thick subcategory of torsion sheaves [Del80, 1.1.1(c)].

The category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves is the $\overline{\mathbb{Q}}_\ell$-linear 2-colimit of categories of the categories of constructible E-sheaves, where $E \subset \overline{\mathbb{Q}}_\ell$ runs over all finite extension of $\mathbb{Q}_\ell$ [Del80, 1.1.1(d)]. A constructible $\overline{\mathbb{Q}}_\ell$-sheaf is said to be lisse if it is locally of the form $\mathcal{F} \otimes_E \overline{\mathbb{Q}}_\ell$, where $\mathcal{F}$ is a lisse sheaf.

(A.1.2) A Weil sheaf $\mathcal{G}_0$ on an algebraic scheme X_0 over $\mathbb{F}_q$ is a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on X together with an isomorphism

$$F^* : F^*(\mathcal{G}) \cong \mathcal{G}$$

where F is the Frobenius automorphism $F : X \to X$.

Following the Deligne's setting, we will simply refer to Weil sheaves as "sheaves" . The constructible $\overline{\mathbb{Q}}_\ell$-sheaves will be called étale sheaves.

(A.1.3) We follow the notation in [Cad18]. If Q is an algebraic extension of $\mathbb{Q}_\ell$, by a Q-coefficient we mean a lisse Weil Q-sheaf.

(A.1.4) We say a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$ of rank 1 on X_0 is of finite order iff there exists $n \geq 1$ such that $\mathcal{C}_0^{\otimes n}$ is trivial.

(A.1.5) Let pr: $X_0 \to \operatorname{Spec}(\mathbb{F}_q)$ be the structural morphism, and let $\mathcal{L}_0$ be a rank 1 $\overline{\mathbb{Q}}_\ell$-coefficient on $\operatorname{Spec}(k)$. For any $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$ on X, the twist of $\mathcal{C}_0$ by $\mathcal{L}_0$ is given by $\mathcal{C}_0 \otimes \operatorname{pr}^* \mathcal{L}_0$. Since $\mathcal{L}_0$ is determined by the image $\alpha \in \overline{\mathbb{Q}}_\ell^\times$, we will also denote $\mathcal{C}_0 \otimes \operatorname{pr}^* \mathcal{L}_0$ as $\mathcal{C}_0^{(\alpha)}$. Note that $\mathcal{L}_0$ is a étale sheaf iff α is ℓ-adic unit.

(A.1.6) The category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves is constructed by taking limits of categories of constructible locally constant sheaves with finite coefficients in characteristic ℓ, which are Galois categories. It follows that for every geometric point $\bar{x}$ lying over $x \in X_0$, the fiber $\mathcal{C}_x$ of a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{C}_0$ is equipped with a continuous action of the étale fundamental group $\pi_1(X_0) := \pi_1(X_0, \bar{x})$ of X_0, and the fiber functor $\mathcal{C}_0 \mapsto \mathcal{C}_x$ induces an equivalence of categories between lisse étale $\overline{\mathbb{Q}}_\ell$-sheaves and finite-dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1(X_0)$. The groups $G(\mathcal{C}), G(\mathcal{C}_0) \subset GL(\mathcal{C}_x)$ respectively identify with the Zariski closure of the images of $\pi_1(X), \pi_1(X_0)$ acting on $\mathcal{C}_x$.

The morphisms $X \to X_0 \to \operatorname{Spec}(\mathbb{F}_q)$ induce a short exact sequence (assuming X_0 is geometrically connected):

$$1 \to \pi_1(X) \to \pi_1(X_0) \to \pi_1(\mathbb{F}_q) \to 1$$

Furthermore, the geometric Frobenius defines a morphism $\mathbb{Z} \to \pi_1(\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$. The category of $\overline{\mathbb{Q}}_\ell$-coefficients can be described in a similar way by replacing the étale fundamental group with the Weil group $W(X_0)$, which is the fiber product $W(X_0) := \pi_1(X_0) \times_{\pi_1(\mathbb{F}_q)} \mathbb{Z}$ equipped with the topology induced by the product of the profinite topology on $\pi_1(X)$ and the discrete topology on $\mathbb{Z}$.

Example (A.1.7) The data of a rank 1 $\overline{\mathbb{Q}}_\ell$-coefficient on $\mathrm{Spec}(\mathbb{F}_q)$ is equivalent to that of an element of $\overline{\mathbb{Q}}_\ell^\times$; the full subcategory of lisse étale $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Spec}(\mathbb{F}_q)$ corresponds to the subgroup $\overline{\mathbb{Z}}_\ell^\times \subset \overline{\mathbb{Q}}_\ell^\times$ of ℓ-adic units.

Proposition (A.1.8) ([Del80, 1.3.4]).

Every rank 1 $\overline{\mathbb{Q}}_\ell$-coefficient on X_0 is the twist of a finite $\overline{\mathbb{Q}}_\ell$-coefficient. In particular, every $\overline{\mathbb{Q}}_\ell$-coefficient on X_0 is the twist of a finite $\overline{\mathbb{Q}}_\ell$-coefficient with finite determinant.

Proof. It follows from the Artin reciprocity law from class field theory. ☐

Proposition (A.1.9) ([Del12, 0.4]).

Let $\mathcal{C}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient on X_0. The following conditions are equivalent:

a) $\mathcal{C}_0$ is étale;

b) $\det(\mathcal{C}_0)$ is étale;

c) $\mathcal{C}_0 = \mathcal{S}_0^{(\alpha)}$, where $\mathcal{S}_0$ is a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf with finite determinant and α is an ℓ-adic unit.

Theorem (A.1.10) (Grothendieck trace formula, [Gro66]).

$$L(X_0, \mathcal{C}_0, T) = \prod_{i\geq 0} \det\left(1 - q^{-d}F^{-1}T \mid H^i(X_0, \check{\mathcal{C}}_0)\right)^{(-1)^{i+1}} = \prod_{i\geq 0} \det\left(1 - FT \mid H^i_c(X_0, \mathcal{C}_0)\right)^{(-1)^{i+1}},$$

where $\check{\mathcal{C}}_0$ is the dual sheaf, and F is the action of geometric Frobenius.

(A.1.11) If $\mathcal{C}_0$ is a $\overline{\mathbb{Q}}_\ell$-coefficient on X_0, then the groups $G(\mathcal{C}), G(\mathcal{C}_0) \subset \mathrm{GL}(\mathcal{C}_x)$ identify respectively with the Zariski closure of the image of $\pi_1(X)$, $W(X_0)$ acting on $\mathcal{C}_x$.

Proposition (A.1.12) (Global monodromy, [Del80, 1.3.8]).

Let $\mathcal{C}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient on X_0. The radical of $G(\mathcal{C})$ is unipotent.

Proof. We have a semidirect exact sequence

$$1 \longrightarrow G(\mathcal{C}) \longrightarrow G(\mathcal{C}_0) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 1,$$

and there exists a positive integer N such that the semisplit sequence

$$1 \longrightarrow G(\mathcal{C}) \longrightarrow \deg^{-1}(N\cdot\mathbb{Z}) \xrightarrow{\deg} N\cdot\mathbb{Z} \longrightarrow 1.$$

is split. Replacing the base field by its degree N extension, we can assume N is 1. Thus we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\bar{x}) \arrow[r] \arrow[d, "\rho"] & \pi_1(X_0,\bar{x}) \arrow[r] \arrow[d, "\rho"] \arrow[dl, dashed] & \mathbb{Z} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G(\mathcal{C}) \arrow[r, shift left] \arrow[d, "\alpha"] & G(\mathcal{C})\times\mathbb{Z} \arrow[r] \arrow[l, shift left, "\pi"] & \mathbb{Z} \arrow[r] & 1 \\
& (G(\mathcal{C}))_{\mathrm{ab}} & & &
\end{tikzcd}$$

By property of reductive group $(G(\mathcal{C}))_{\mathrm{ab}}$ is the product of a finite group and a torus. Conisder the composition $\alpha \circ \pi \circ \rho$, we can show that the image of $\pi_1(X_0, \bar{x})$ in $(G(\mathcal{C}))_{\mathrm{ab}}$ is finite. But it must be Zariski dense by definition of $G(\mathcal{C})$, so $(G(\mathcal{C}))_{\mathrm{ab}}$ is finite and hence $G(\mathcal{C})$ is semi-simple. By passing to the identity component, we get that $G(\mathcal{C})$ is unipotent.

□

Let $\mathbb{F}_q$ be a finite field, $\mathbb{F} = \overline{\mathbb{F}}_q$ and let X_0 be a normal scheme of finite type over $\mathbb{F}_q$. Let ℓ and ℓ' be prime numbers not dividing q. For brevity, we take the definition used in [Ked22] and [Cad18]: if Q is an algebraic extension of $\mathbb{Q}_\ell$, by a Q-coefficient $\mathcal{C}_0$ we mean a lisse Weil Q-sheaf. Let $\mathcal{C}$ be its base change to $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

(A.1.13) Let $\chi_x(\mathcal{C}, T) := \det(1 - F_x T, \mathcal{C}_0) \in Q[T]$ be the caracteristic polynomial at a closed point x. The rank r of $\mathcal{C}_x$ is independent of x, and we have a characteristic polynomial mapping:

$$\begin{aligned} \chi_-(\mathcal{C}) : |X_0| \rightarrow \quad & \mathcal{P}_r\left(\overline{\mathbb{Q}}_\ell\right) \\ x \quad \rightarrow \quad & \chi_x(\mathcal{C}, T) \end{aligned}$$

takes values in the $\overline{\mathbb{Q}}_\ell$-points of the $\mathbb{Q}$-variety $\mathcal{P}_r := \mathbb{G}_m \times \mathbb{A}^{r-1}$ of degree-r polynomials with constant term 1. The morphism $(\mathbb{G}_m)^r \rightarrow \mathcal{P}_r$ that takes $(\beta_1, \ldots, \beta_r)$ to the polynomial $\prod_{i=1}^r (1 - \beta_i t)$ induces an isomorphism $(\mathbb{G}_m)^r / S_r \xrightarrow{\sim} \mathcal{P}_r$, where S_r is the symmetric group.

(A.1.14) For a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_0$, we say that $\mathcal{C}_0$ is algebraic if for all $x \in |X_0|$, $\chi_x(\mathcal{C}, T)$ has coefficients in the field of algebraic numbers. Given a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}_1$ and a $\overline{\mathbb{Q}}_{\ell'}$-coefficient $\mathcal{C}_2'$ and fixed an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \overline{\mathbb{Q}}_{\ell'}$, we say they are companions if for all closed points in X_0, the characteristic polynomials of Frobenius on $\mathcal{C}_{1,x}$ and $\mathcal{C}_{2,x}$ coincide.

Proposition (A.1.15) Let $\mathcal{C}_1, \mathcal{C}_2$ be two algebraic $\overline{\mathbb{Q}}_\ell$-coefficients on X_0 which are companions, then

(a) If $\mathcal{C}_1$ is irreducible, then so is $\mathcal{C}_2$.

(b) $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same semi-simplification.

Proof. In both cases, we may assume that X_0 is irreducible of pure dimension d, and by [Del80, 1.8.10] that $\mathcal{C}_1, \mathcal{C}_2$ are both pure of weight 0 and semi-simple.

In case (a), note that by Schur's lemma, as a $W(X_0, x)$-module, $\mathcal{C}_i$ is irreducible if and only if $H^0\left(X, \mathcal{C}_i^\vee \otimes \mathcal{C}_i\right)^F$ is one-dimensional. We may thus apply Lemma (A.2.6) to $\mathcal{C}_1^\vee \otimes \mathcal{C}_1, \mathcal{C}_2^\vee \otimes \mathcal{C}_2$ to conclude.

In case (b), it suffices to check that any irreducible subobject $\mathcal{F}$ of $\mathcal{C}_1$ (which must also be pure of weight 0) also occurs as a summand of $\mathcal{C}_2$. To this end, by Schur's lemma again, note that $\mathcal{F}$ occurs as a summand of $\mathcal{C}_i$ if and only if $H^0\left(X, \mathcal{F}^\vee \otimes \mathcal{C}_i\right)^\varphi \neq 0$; we may thus apply Lemma (A.2.6) to $\mathcal{C}_1^\vee \otimes \mathcal{F}, \mathcal{C}_2^\vee \otimes \mathcal{F}$ to conclude. □

(A.1.16) Let K be a global field, $\overline{K}$ denote a separable closure of K. In [Ser98] Serre defined strictly compatible system of ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$:

Let S be a finite set of non-archimedean primes of K. A compatible system is consisted of a continuous representation ρ_ℓ of $\mathrm{Gal}(\overline{K}/K)$ on a finite dimensional $\mathbb{Q}_\ell$-vector space V_ℓ, for all $\ell \nmid \mathrm{char}(K)$. One assumes that ρ_ℓ is unramified at every non-archimedean place $v \notin S$ whose residue characteristic is not ℓ. For all such ℓ, v, the characteristic polynomial of the image $\rho_\ell\left(\mathrm{Frob}_v\right)$ of Frobenius is well-defined, and the compatibility condition states that its coefficients lie in $\mathbb{Q}$ and depend only on v. Clearly, this condition implies that the dimension n of V_ℓ is independent of ℓ. We assume that the system is pure of weight $w \in \mathbb{Z}$, i.e. that the eigenvalues of $\rho_\ell\left(\mathrm{Frob}_v\right)$ have absolute value $q_v^{w/2}$ for every complex embedding, where q_v is the number of elements in the residue field of v.

Example (A.1.17) Let K is a function field. Let $\mathbb{F}_q \subset K$ be its field of constants and let X_0 be smooth geometrically connected algebraic curve over $\mathbb{F}_q$ with function field K.

We remove from X_0 the finite set where the ρ_ℓ may be ramified, and fix a geometric point $\bar{x}$ of X_0. Then each ρ_ℓ comes from a representation of the étale fundamental group $\pi_1(X_0, \bar{x})$ which we denote again by ρ_ℓ. Every V_ℓ is the stalk at $\bar{x}$ of a lisse ℓ-adic sheaf $\mathcal{F}_0$ on X_0, which is pointwise pure of weight w.

The Zariski closure of $\rho_\ell(\pi_1(X, \bar{x}))$ is the geometric fundamental group $G(\mathcal{F}_0)$ (A.1.11) is semisimple(A.2.7).

Proposition (A.1.18) The dimension of the space of invariants $V_\ell^{G(\mathcal{F}_0)}$ is independent of ℓ.

Proof. As a curve, the cohomology with compact support $H_c^i(X, \mathcal{F})$ vanishes in degrees $i > 2$.

For $i = 2$ $H_c^i(X, \mathcal{F})$ is canonically isomorphic to $V_\ell^{G(\mathcal{F}_0)}(-1)$, where (-1) denotes Tate twist; it is therefore pure of weight $w + 2$.

In degrees $i = 1$, by [Del80, 3.3.1] $H_c^i(X, \mathcal{F})$ has weights $< w + 2$.

It follows that the dimension in question can be described as the sum of the multiplicities of all Frobenius eigenvalues of weight $w+2$ in the virtual representation $\sum(-1)^i H_c^i(X, \mathcal{F})$. By the Lefschetz trace formula (A.1.10) this number depends only on the zeta function of $(X_0, \mathcal{F}_0)$, which is, by the compatibility assumption, independent of ℓ. $\square$

A.2 Weights Theory

Fix ι an embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, $\mathcal{C}_0$ a $\overline{\mathbb{Q}}_\ell$-coefficient.

Definition (A.2.1)

1. For $x \in |X_0|$, the multiset of ι-weights of $\mathcal{C}_0$ at x is the multiset consisting of $-2\log_{\#\kappa(x)} |\iota(\lambda)|$ as λ varies over the roots of $\chi_x(\mathcal{C}, T)$ (counted with multiplicity).
2. $\mathcal{C}_0$ is ι-pure of weight w if for all $x \in |X_0|$, the multiset of ι-weights of $\mathcal{C}_0$ at x consists of the single element w.
3. $\mathcal{C}_0$ is ι-mixed if it is a successive extension of $\overline{\mathbb{Q}}_\ell$-coefficients that are ι-pure.

Remark (A.2.2) The definition for ι-mixed differs from the definiton given in [Del80, 1.2.2 (ii)]. But they coincides [Del80, 1.8.11] by the semicontinuity of weight [Del80, 1.8.10]. Notice that we don't require X_0 to be normal by [Del80, 3.4.11].

Proposition (A.2.3) (semicontinuity of weight) Suppose that for some $w \in \mathbb{R}$, there exists an open dense subset U_0 of X_0 (not necessary normal) such that $\mathcal{C}_0|_{U_0}$ is ι-pure of weight w. Then $\mathcal{C}_0$ is also ι-pure of weight w. What's more, if X_0 is irreducible and normal, $\mathcal{C}_0$ is irreducible and $\mathcal{C}_0|_{U_0}$ is ι-mixed, then $\mathcal{C}_0$ is ι-pure.

Proof. For the first argument, see [Del80, Corollary 1.8.9], use the local monodromy of pure sheaves and dévissage. On the other hand, we can deduce it immediately from (A.2.4). For the second argument, notice that if X_0 is normal, we have $\pi_1(U_0) \twoheadrightarrow \pi_1(X_0)$, thus if $\mathcal{C}_0$ is irreducible, $\mathcal{C}_0|_{U_0}$ is irreducible, so $\mathcal{C}_0|_{U_0}$ is ι-pure, then apply the first argument. $\square$

Proposition (A.2.4) The multiset of ι-weights of $\mathcal{C}_0$ at $x \in |X_0|$ is independent of x.

Proof. See [Del80, Corollary 1.8.12]. On the other hand, we can also use [Laf02, VII.6]. It suffices to compare two points $x, y \in |X_0|$. By restricting to a curve in X_0 through x, y, we may assume that X_0 is a curve (such a curve exists by Hilbert irreducibility, see [EK12, Appendix B]); in this case, we may further assume that $\mathcal{C}_0$ is irreducible and its determinant is of finite order by a twist. We may then apply [Laf02, VII.6] to deduce that $\mathcal{C}_0$ is ι-pure of weight 0. $\square$

Theorem (A.2.5) ([Del80, 3.3.4]).

Let $f\colon X_0 \to Y_0$ be a finite type morphism of schemes over $\mathbb{Z}$. If $\mathcal{C}_0$ is ι-pure of weight w, then for each i, $R^i f_! \mathcal{C}_0$ is mixed of weight $\leq n+i$.

Sketch of proof: The main theorem in [Del80]. For a sketch of proof, using dévissage and a generic argument, we reduce to the case X_0 is the complement in $\overline{X}_0$, a smooth and projective scheme of pure relative dimension 1 over Y_0, of an étale finite divisor D_0 over Y_0, and that $\mathcal{C}_0$ is tamely ramified along D_0. For the curve case, we use Rankin-Selberg method, Lefschetz pencil from [SGA7-II] and several results of weights theory built in [Del80]: τ-real sheaves, semicontinuity of weights, and monodromy. To use Rankin-Selberg method, we make induction, and the key trick is to apply Lefschetz pencil to the tensor product of an open dense subset U_0 and use Künneth formula, cf. [Del80, 3.2.6].

$\square$

Corollary (A.2.6) Suppose that X_0 is irreducible of pure dimension d and that $\mathcal{C}_0$ is a ι-pure of weight 0 $\overline{\mathbb{Q}}_\ell$-coefficient. Then the pole order of $L(\mathcal{C}, T)$ at $T = q^{-d}$ equals the multiplicity of 1 as an eigenvalue of Frobenius on $H^0(X, \mathcal{C}^\vee)$.

Proof. By the Lefschetz trace formula (A.1.10), the factor $i = 0$ contributes the predicted value to the pole order. Meanwhile, by (A.2.5), the eigenvalues of F on $H^i(X, \mathcal{E}^\vee)$ for $i > 0$ all have absolute value at least $q^{1/2}$, so the corresponding factor of (A.1.10) only has zeroes or poles in the region $|T| \geq q^{-d+1/2}$. This proves the claim. $\square$

Corollary (A.2.7) If $\mathcal{C}_0$ is a pure $\overline{\mathbb{Q}}_\ell$-coefficient on X_0, then $\overline{\mathcal{C}}$ is semi-simple; in particular, $G(\overline{\mathcal{C}})$ is semi-simple.

Proof. See [Del80, 3.4.1(iii)]. The key is from the Hochschild–Serre spectral sequence we get a exact sequence like

$$H^0(X, \mathcal{C}_2^\vee \otimes \mathcal{C}_1)_F \to \mathrm{Ext}(\mathcal{C}_2, \mathcal{C}_1) \to H^1(X, \mathcal{C}_2^\vee \otimes \mathcal{C}_1)^F,$$

and by the Deligne's purity theory [Del80, Theorem 3.3.1] the Frobenius invariant part can only be null. $\square$

Use the proof quite similar to (A.2.7) (compare weights from two parts), Deligne gives another two aplications: local invariant cycle theorem [Del80, Theorem 3.6.1] and hard Lefschetz theorem [Del80, 4.1].

(A.2.8) If V is an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $W(X_0, x)$, its restriction to $\pi_1(X, x)$ is a sum of non-isomorphic irreducible representations of $\pi_1(X, x)$ by Clifford's theorem, permuted transitively by $W(X_0, x)/\pi_1(X, x) = \mathbb{Z}$.

Let n be the number of summands, and $X_1 := X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$; $W(X_1, x)$ is the inverse image in $W(X_0, x)$ of $n\mathbb{Z}$. If S is one of the irreducible summands of the restriction of V to $\pi_1(X, x)$, use the proof of Clifford's theorem it's easy to see that S is a representation of $W(X_1, x)$ and V is induced from $\mathrm{Ind}_{W(X_1,x)}^{W(X_0,x)}(S)$.

If $\mathcal{V}$ (resp. $\mathcal{S}$) is the corresponding $\overline{\mathbb{Q}}_\ell$-Weil sheaf for V (resp. S), $\mathcal{V}$ is the direct image, via $X_1 \to X_0$, of $\mathcal{S}$.

(A.2.9) Now let V be a semisimple representation of $W(X_0, x)$. The quotient $\mathbb{Z}$ of $W(X_0, x)$ permutes the isomorphism classes of simple constituents of the restriction of V to $\pi_1(X, x)$ by (A.2.8).

Let A be the set of orbits, and in each orbit a, choose a representative S'_a. Let $n(a)$ be the number of elements in the orbit. If $X_a = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n(a)}}$, the representation S'_a extends to a representation S_a of $W(X_a, x)$, and S_a is unique up to torsion by a character of the quotient $\mathbb{Z}$ of $W(X_a, x)$. We choose the torsion so that the character $\det(S_a)$ of $W(X_a, x)$ is of finite order.

Applying (A.2.8) to the irreducible constituents of V and grouping the occurrences of S_a, we obtain a decomposition

$$V = \bigoplus\nolimits_a \operatorname{Ind}_{W(X_a,x)}^{W(X_0,x)} (S_a \otimes W_a)$$

for W_a a representation of $W(X_a, x)$ trivial on $\pi_1(X, x)$. This description of V is unique, up to the following:

a) twisting S_a by a finite-order character of $\mathbb{Z} = W(X_a, x)/\pi_1(X, x)$, and W_a by the inverse character.

b) replacing S_a with its conjugate $S_a^{(i)}$ by an element i of $W(X_0, x)/W(X_a, x) = \mathbb{Z}/n(a)$.

(A.2.10) Let's switch from the language of group representations to that of sheaves: S_a (resp. $S_a^{(i)}$) corresponds to $\mathcal{S}_{a,1}$ (resp. $\mathcal{S}_{a,1}^{(i)}$) on X_a, W_a corresponds to $\mathcal{W}_a$ on $\operatorname{Spec}(\mathbb{F}_{q^{n(a)}})$, and if p_a is the projection from X_a to X_0, and pr_a is the projection from X_a to $\operatorname{Spec}(\mathbb{F}_{q^{n(a)}})$, then (A.2.9) becomes a decomposition of the Weil $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{V}$ on:

$$\mathcal{V} = \bigoplus_{a \in A} p_{a*} (\mathcal{S}_{a,1} \otimes \operatorname{pr}_a^* \mathcal{W}_a). \tag{A.1}$$

In this decomposition, the $\det(\mathcal{S}_a)$ are of finite order, and the inverse images $\mathcal{S}_a^{(i)}$ on X of the $\mathcal{S}_{a,1}^{(i)}$ (for $a \in A$ and $i \in \mathbb{Z}/n(a)$) are irreducible $\overline{\mathbb{Q}}_\ell$-sheaves that are not isomorphic to each other.

Proposition (A.2.11) ([Del12, 1.3, 1.4])

Let $\mathcal{C}_0$ be a $\overline{\mathbb{Q}}_\ell$-coefficient. For $n \geq 1$, let k_n be the extension of degree n of $k = \mathbb{F}_q$ in $\bar{k}$, and let $p_n : X_{k_n} \to X_0$ be the canonical projection. Then $\mathcal{C}_0|_{X_{k_n}}$ is irreducible if and only if $\mathcal{C}_0|_X$ is irreducible for every $n \geq 1$.

Proof. The sufficiency is trivial. Necessity follows from the discussion above. ☐

B Ramification at Infinity

B.1 Recall of SGA7 XIII.2

(B.1.1) Let X be an S-scheme and D be a effective divisor on X. Recall [SGA5, II 4.2] that we say D has strict normal crossings relative to S if there exists a finite family $(f_i)_{i \in I}$ of elements in $\Gamma(X, \mathscr{O}_X)$ such that $D = \sum_{i \in I} \operatorname{div}(f_i)$ and the following condition is satisfied: for every point x in $\operatorname{Supp} D$, X is smooth over S at x, and if we denote $I(x)$ as the set of $i \in I$ such that $f_i(x) = 0$, then the subscheme $V\left((f_i)_{i \in I(x)}\right)$ is smooth over S and has codimension $\operatorname{card} I(x)$ in X.

The divisor D is said to have normal crossings relative to S if, locally on X in the étale topology, it has strict normal crossings.

(B.1.2) Let D be a divisor with normal crossings relative to S. We define $Y = \operatorname{Supp} D$, $U = X - Y$, and denote by $i : U \to X$ the canonical immersion. For every geometric point $\bar{s}$ of S and every maximal point y of the geometric fiber $Y_{\bar{s}}$, the ring $R = \mathscr{O}_{X_{\bar{s}}, y}$ is a discrete valuation ring.

Definition (B.1.3) Let F be a sheaf of sets on U. We say that F is tamely ramified on X (along D) relative to S if, for every geometric point $\bar{s}$ of S, the following condition is satisfied: for every maximal point y of $Y_{\bar{s}}$, the restriction of F to the fraction field K of $\mathscr{O}_{X_{\bar{s}}}$ is represented by the spectrum of an étale K-algebra L which is tamely ramified over $\mathscr{O}_{X_{\bar{s},y}}$.

Definition (B.1.4) If F is a sheaf of groups on U, tamely ramified on X relative to S, we denote by $H^1_{\mathrm{t}}(U, F)$ the subset of $H^1(U, F)$ consisting of classes of (left) torsors under F that are tamely ramified on X relative to S.

(B.1.5) Let $C_t((U, X)/S)$ or simply C_t be the category of étale coverings of U that are tamely ramified on X relative to S.

Suppose U is connected, and let a be a geometric point of U. Let Γ_t be the functor that assigns, to an étale covering U' of U that is tamely ramified on X relative to S, the set of geometric points of U' lying over a. The functor Γ_t is represented by a pro-object, which is called the tamely ramified universal covering of (U, X) relative to S, punctured at a. The group opposite to the group of U-automorphisms of the tamely ramified universal covering is called the tame fundamental group and is denoted by $\pi_1^{\mathrm{t}}((U, X)/S, a)$ or simply $\pi_1^{\mathrm{t}}(U, a)$ or even $\pi_1^{\mathrm{t}}(U)$. We have a canonical surjective map $\pi_1(U, a) \twoheadrightarrow \pi_1^{\mathrm{t}}(U, a)$ which is an isomorphism if X is proper (cf. [SGA7-II, V 6.9]).

The importance of the notion of a normal crossings divisor is its role in:

Lemma (B.1.6) (Abhyankar's Lemma,[SGA1, XIII, Prop 5.2, Cor 5.3]).

Let Y be a regular noetherian scheme, X a normal noetherian scheme, and $f\colon X \to Y$ a finite, flat, generically étale map which is tamely ramified. If the support of the branch scheme $\mathscr{B}_{X/Y}$ of f, the closed subscheme defined by the annihilator of $f_*\Omega^1_{X/Y}$ on Y, coincides with the support of a normal crossings divisor D on Y, then

1. X is regular,
2. $\mathscr{B}_{X/Y} = D$ as closed subschemes of Y, so $\mathscr{B}_{X/Y}$ is a normal crossings divisor on Y,
3. for each $y \in \mathscr{B}_{X/Y}$ and $x \in f^{-1}(y)$, there is an isomorphism of $\mathscr{O}^{\mathrm{sh}}_{Y,y}$-algebras

$$\mathscr{O}^{\mathrm{sh}}_{X,x} \simeq \mathscr{O}^{\mathrm{sh}}_{Y,y}[T_1, \dots, T_r] / (T_1^{e_1} - f_1, \dots, T_r^{e_r} - f_r)$$

where $f_1, \dots, f_r$ define the normal crossings divisor D in an étale neighborhood of y and $e_1, \dots, e_r \geq 1$ are relatively prime to the characteristic of $k(y)$.

B.2 Swan Conductor

(B.2.1) Let X be a smooth k-curve and $\bar{x}$ a geometric point above $x \in |X_0|$. Let $U := X \backslash x$ be the complement of x in X, $X_{(x)} := \mathrm{Spec}(\mathscr{O}^h_{X,x})$ be the spectrum of the henselianization of the local ring $\mathscr{O}_{X,x}$ at x, and $X_{(\bar{x})} := \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ be the spectrum of the strict henselization defined by $\bar{x}$. Let $U_{(x)} := U \times_X X_{(x)}$ and $U_{(\bar{x})} := U \times_X X_{(\bar{x})}$. Let $I_x := \pi_1(U_{(\bar{x})}) \subset D_x := \pi_1\left(U_{(x)}\right)$ denote the inertia and decomposition groups of X at x. We have a short exact sequence that splits:

$$1 \to I_x \to D_x \to \pi_1(x) \to 1,$$

And we denote by $P_x \subset I_x$ the unique p-Sylow subgroup of I_x (the wild inertia group) and by $I_x^t := I_x/P_x$ (the tame inertia group). There exists a $\pi_1(k)$-equivariant isomorphism $I_x^t \xrightarrow{\sim} (\widehat{\mathbb{Z}}/\mathbb{Z}_p)(-1)$.

(B.2.2) Let $\mathcal{C}$ be a $\overline{\mathbb{Q}}_\ell$-coefficient on X. To $\mathcal{C}|_{U(x)}$ is associated a representation of D_x on a finite-dimensional Q-vector space $\mathcal{C}_{\bar{x}}$. This allows us to define the local Swan conductor $\mathrm{Sw}_x(\mathcal{C})$ of $\mathcal{C}$ at x. The group D_x is equipped with a decreasing filtration $I_x^{(\lambda)}, \lambda \geq 0$ by closed normal subgroups (higher ramification subgroups-[Ser80, S68, Chap. IV]) such that:

$-\bigcap_{\lambda'<\lambda} I_x^{(\lambda')} = I_x^{(\lambda)}$, $\bigcap_{\lambda_x \in \mathbb{R}} I_x^{(\lambda)} = 0$,

$-I_x := I_x^{(0)} \subset D_x$ is the inertia group,

$-P_x := I_x^{(0+)} \subset I_x^{(0)}$ is the wild inertia group, i.e., the p-Sylow subgroup of $I_x^{(0)}$, where we define $I_x^{(\mu+)} := \cup_{\lambda<\mu} I_x^{(\lambda)} \subset I_x^{(\mu)}$.

Let us denote $\mathcal{C}_x^{ss}$ as the I_x-semisimplification of $\mathcal{C}_x$. If $W \subset \mathcal{C}_x^{ss}$ is a nontrivial simple submodule of P_x, there exists a unique $\lambda > 0$ such that $W^{I_x^{(\lambda+)}} = 0$ and $W^{I_x^{(\lambda)}} \neq 0$. We call λ the slope of W, and $(\mathcal{C}_x^{ss})^{I_x^{(\lambda+)}}/(\mathcal{C}_x^{ss})^{I_x^{(\lambda)}}$ is the sum of simple P_x-submodules of $\mathcal{C}_x^{ss}$ with slope λ. With these notations, we define

$$\mathrm{Sw}_x(\mathcal{C}) = \sum_{\lambda>0} \lambda \dim\left((\mathcal{C}_x^{ss})^{I_x^{(\lambda+)}}/(\mathcal{C}_x^{ss})^{I_x^{(\lambda)}}\right).$$

(B.2.3) If $\mathrm{Sw}_x(\mathcal{C}) = 0$, we say that $\mathcal{C}$ is tamely ramified at x. We also denote the slope λ of $\mathcal{C}$ at x as $\alpha(\mathcal{C}_x) = \alpha_x(\mathcal{C})$ which $\mathcal{C}_x$ is the corresponding Galois representation.

(B.2.4) If the representation $\mathcal{C}_x$ is irreducible, by the definition we have

$$\mathrm{Sw}_x(\mathcal{C}) = \dim(\mathcal{C}_x) \cdot \alpha(\mathcal{C}_x) \qquad (V \text{ irreducible}).$$

Then for $\mathcal{C}_x$ not necessarily irreducible, by additivity of Swan conductor, we have

$$\mathrm{Sw}_x(\mathcal{C}) \leq \dim(\mathcal{C}_x) \cdot \alpha(\mathcal{C}_x). \tag{B.1}$$

(B.2.5) Let X be a smooth curve over k which is algebraically closed, $X \hookrightarrow \overline{X}$ its smooth compactification, and $\mathcal{C}$ a $\mathbb{Q}_\ell$-coefficient on X. The local Swan conductors are related to the Euler-Poincaré characteristic $\chi_c(\mathcal{C}) := \sum_{i\geq 0}(-1)^i \dim H_c^i(X,\mathcal{C})$ through the Grothendieck-Ogg-Shafarevich formula:

Theorem (B.2.6) ([Lau87, Theorem 2.2.1.2]).

$$\chi_c(\mathcal{C}) = \mathrm{rank}(\mathcal{C})\chi_c(X) - \sum_{x\in\overline{X}\setminus X} n(x)\,\mathrm{Sw}_x(\mathcal{C}).$$

Sometimes it is convenient to globalize the definition of the Swan conductor by considering the effective divisor $\mathrm{Sw}(\mathcal{C}) = \sum_{x\in\overline{X}\setminus X} \mathrm{Sw}_x(\mathcal{C})[x]$, $X \subset \overline{X}$ is a smooth compactification.

Definition (B.2.7) Let $D \in \mathrm{Div}^+(\overline{X})$ be an effective Cartier divisor. The subset $\mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D) \subset \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X)$ is defined by the condition $\mathrm{Sw}(V) \leq D$. If V lies in $\mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)$, we say that its ramification is bounded by D.

(B.2.8) It's shown in [EK12, Proposition 3.9] that for any $V \in \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X)$ there is a divisor D with $V \in \mathcal{C}_{\overline{\mathbb{Q}}_\ell,r}(X,D)$.

Example (B.2.9) If X is an affine curve, then we have $H_c^0(X,\mathcal{C}) = 0$, then by (B.2.6) and (B.1), we get

$$\dim H_c^1(X,\mathcal{C}) \leq r\left(b_1(X) + \sum \alpha_x(\mathcal{C})\right). \tag{B.2}$$

B.3 Dimension ≥ 2

In higher dimensions, there are several possible definitions of the notion of tameness at infinity. Let $X \hookrightarrow \overline{X}$ be a normal compactification. We say that a $\overline{\mathbb{Q}}_\ell$-coefficient $\mathcal{C}$ on X is tamely ramified at a point $x \in \overline{X}\setminus X$ of codimension

1 if the representation of $\pi_1\left(X_{(\bar{x})}\right)$ corresponding to $\mathcal{C}|_{X_{(\bar{x})}}$ is. We say that $\mathcal{C}$ is tame along $\overline{X}\backslash X$ if it is tame at every point $x \in \overline{X}\backslash X$ of codimension 1.

When X is smooth over k, the following conditions:

1. (curve)-tameness: For every smooth curve C over k and every morphism $C \to X$, $\mathcal{C}|_C$ is tame.

2. (divisor)-tameness: For every normal compactification $X \hookrightarrow \overline{X}$, $\mathcal{C}$ is tamely ramified along $\overline{X}\backslash X$.

are equivalent, and we will simply say that $\mathcal{C}$ is tame.

When X admits a smooth compactification $X \hookrightarrow \overline{X}$ such that $\overline{X}\backslash X$ is a normal crossings divisor, $\mathcal{C}$ is tame if and only if it is tame along $\overline{X}\backslash X$.

References

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers (I). CIRM, 6 - 10 juillet 1981*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982. Luminy conference ; Conference date: 06-07-1981 Through 11-07-1981.

[Cad18] A. Cadoret. A Tannakian Cebotarev density theorem and companions. - imj-prg, 2018. URL `https://webusers.imj-prg.fr/~anna.cadoret/TannakianCebo.pdf`.

[Del77] P. Deligne. *Cohomologie étale*. Number 569 in Lecture Notes in Mathematics. Springer-Verlag, 1977.

[Del80] P. Deligne. La conjecture de Weil : II. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980. URL `http://www.numdam.org/item/PMIHES_1980__52__137_0/`.

[Del12] P. Deligne. Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie. *Moscow Mathematical Journal*, 12:497–514, 01 2012. doi:10.17323/1609-4514-2012-12-3-497-514.

[Dri18] V. Drinfeld. On a conjecture of Deligne, 2018, 1007.4004.

[EK11] H. Esnault and M. Kerz. Notes on Deligne Letter to Drinfeld. 2011.

[EK12] H. Esnault and M. Kerz. A finiteness theorem for Galois representations of function fields over finite fields (after Deligne), 2012, 1208.0128.

[Gro66] A. Grothendieck. Formule de Lefschetz et rationalité des fonctions *L*. *Séminaire Bourbaki*, 9:41–55, 1964-1966. URL `http://eudml.org/doc/109708`.

[Jan92] U. Jannsen. Motives, numerical equivalence, and semi-simplicity. *Inventiones Mathematicae*, 107(1):447–452, Dec. 1992. doi:10.1007/bf01231898.

[Jon96] A. J. Jong. Smoothness, semi-stability and alterations. *Publications mathématiques de l'IHÉS*, 83(1):51–93, Dec. 1996. doi:10.1007/bf02698644.

[Kat01] N. M. Katz. Sums of Betti numbers in arbitrary characteristic. *Finite Fields and Their Applications*, 7(1):29–44, 2001. doi:https://doi.org/10.1006/ffta.2000.0303.

[Ked22] K. S. Kedlaya. Etale and crystalline companions, I, 2022, 1811.00204.

[KM74] N. M. Katz and W. Messing. Some consequences of the riemann hypothesis for varieties over finite fields. *Inventiones Mathematicae*, 23(1):73–77, Mar. 1974. doi:10.1007/bf01405203.

[KS09a] M. Kerz and A. Schmidt. Covering data and higher dimensional global class field theory, 2009, 0804.3419.

[KS09b] M. Kerz and A. Schmidt. On different notions of tameness in arithmetic geometry, 2009, 0807.0979.

[Laf02] L. Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Inventiones mathematicae*, 147(1):1–241, Jan. 2002. doi:10.1007/s002220100174.

[Lau87] G. Laumon. Transformation de Fourier constantes d'équations fonctionnelles et conjecture de Weil. *Publications mathématiques de l'IHÉS*, 65(1):131–210, Dec. 1987. doi:10.1007/bf02698937.

[MO] P. Mannisto and M. Olsson. Independence of ℓ and surfaces. URL `https://math.berkeley.edu/~molsson/surfaces.pdf`.

[Poo04] B. Poonen. Bertini theorems over finite fields. *Annals of Mathematics*, 160(3):1099–1127, nov 2004. doi:10.4007/annals.2004.160.1099.

[Ser80] J.-P. Serre. *Corps locaux.* Number 1296 in Actualités scientifiques et industrielles. Hermann, Paris, 3. éd., corr edition, 1980.

[Ser98] J. Serre. *Abelian ℓ-adic Representations and Elliptic Curves.* Research Notes in Mathematics (A K Peters), Vol 7. Peters, 1998. URL `https://books.google.fr/books?id=CLlkdN2lh8cC`.

[Ser16] J.-P. Serre. *Lectures on $N_X(p)$.* A K Peters/CRC Press, Apr. 2016. doi:10.1201/b11315.

[SGA1] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1),1971*, volume 224 of *Lecture notes in mathematics.* Springer-Verlag, 1972.

[SGA2] A. Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), volume 2 of *Advanced Studies in Pure Mathematics.* North-Holland Publishing Co., 2. Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962.

[SGA4] M. Artin, A. Grothendieck, and J.-L. Verdier. *Theorie des Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics.* Springer, 1971.

[SGA5] A. Grothendieck. *Séminaire de géométrie algébrique du Bois-Marie 1965-66, Cohomologie l-adique et fonctions L, SGA5*, volume 589 of *Springer Lecture Notes.* Springer-Verlag, 5.

[SGA7-II] P. Deligne and N. Katz. *Groupes de monodromie en géométrie algébrique. II.* Lecture Notes in Mathematics, Vol 340. Springer-Verlag, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7, II).

[Stacks] Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`, 2025.