Notes on Artin's Theorem on Elementary Fibrations in SGA 4

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Definition [SGA4, Définition 3.1.]

An elementary fibration is a morphism of schemes $f\colon X \to S$ that can be embedded into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] & Y \arrow[l, "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

1. j is a dense open immersion in each fiber, and $X = \overline{X} \setminus Y$.
2. $\bar{f}$ is smooth and projective, with geometrically irreducible fibers of dimension 1.
3. g is an étale covering, and each fiber of g is non-empty.

Definition [SGA4, Définition 3.2.]

We call a good neighborhood relative to S an S-scheme X such that there exist S-schemes

$$X = X_n, \ldots, X_0 = S$$

and elementary fibrations $f_i : X_i \to X_{i-1}$, $i = 1, \ldots, n$.

Theorem [SGA4, Proposition 3.3]

Let k be an algebraically closed field, X a smooth scheme over $\operatorname{Spec}(k)$, and $x \in X$ a rational point. There exists an open subset of X containing x that is a good neighborhood (relative to $\operatorname{Spec}(k)$).

Proof. We may assume that X is irreducible. By induction on $\dim X = n$, it suffices to find a neighborhood U of x in X and an elementary fibration $f\colon U \to V$, where V is smooth of dimension $n-1$. Indeed, there will exist a neighborhood V' of $v = f(x)$ that is a good neighborhood, and we can take $U' = U \cap V'$ as a good neighborhood of x.

Since the theorem is Zariski local, we may suppose that $X \subset \mathbb{A}^r$ is affine. Let X_0 be the closure of X in $\mathbf{P}^r$, and we can write $X_0 = X \cup D$ with D ample. Let $\overline{X}$ be the normalization of X_0 and $\pi : \overline{X} \to X_0$, the preimage $\pi^{-1}D$ is ample with complement X_0 because π is finite. Set $Y = \overline{X} \setminus X$, with the reduced induced structure. Furthermore, let $S \subset \overline{X}$ denote the closed subset of singular points. We have $S \subset Y$, and:

$$\begin{aligned} &\dim \overline{X} = \dim X = n, \\ &\dim Y = n-1 \quad [\text{Stacks}, \mathit{Tag0BCV}], \\ &\dim S \le n-2 \quad (\text{since normal implies smooth in codimension 1}), \end{aligned}$$

Embed $\overline{X}$ into a projective space $\mathbf{P}^N$ using the ample bundle. There exist hyperplanes $H_1, \ldots, H_{n-1}$ in $\mathbf{P}^N$, where H_i is the zero set of:

$$\sum_{v=0}^{N} a_{iv} x_v = 0,$$

such that H_i contains x and the intersection $L = H_1 \cap \cdots \cap H_{n-1}$ has dimension $N - n + 1$ and intersects $\overline{X}$ and Y transversely. By Bertini's theorem, the intersection $\overline{X} \cap L$ is a smooth and connected curve, and $Y \cap L$ is of dimension 0 so is a finite set.

Consider the projection $\mathbf{P}^N \to \mathbf{P}^{n-1}$ defined using the projective coordinates:

$$y_i = \sum_{v=0}^{N} a_{iv} x_v.$$

This is a rational map defined outside the projection center $C = H_0 \cap \cdots \cap H_{n-1}$. Let $\epsilon : P' \to \mathbf{P}^N$ be the blow-up of C, giving a diagram

$$\begin{tikzcd} \mathbf{P}^N \arrow[r, "\epsilon"] \arrow[dr] & P' \arrow[d, "\pi"] \\ & \mathbf{P}^{n-1} \end{tikzcd}$$

where π is a morphism [Lü93, Lemma 2.2]. Let $\overline{X}' \subset P'$ denote the strict transform, i.e., the closure of $\epsilon^{-1}(\overline{X} \backslash (\overline{X} \cap C))$.

Since C intersects $\overline{X}$ transversely, the morphism $\overline{X}' \to \overline{X}$ is the blow-up of the finite set $\overline{X} \cap C$. Let $X' = X \backslash (\overline{X} \cap C)$, which also identifies as an open subscheme of $\overline{X}'$, and let $Y' = \overline{X}' \backslash X'$ be the closed subscheme with reduced induced structure. We have the following diagram $\mathbf{D}$ of morphisms:

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[dr, "f'"'] & \overline{X}' \arrow[d, "\overline{f}'"] & Y' \arrow[l, "i"'] \arrow[dl, "g'"] \\ & \mathbf{P}^{n-1} & \end{tikzcd}$$

Finally, we claim that there exists a neighborhood V of $v = f'(x)$ such that the restriction of $\mathbf{D}$ to V satisfies the condition of elementary fibration. Condition 1 is obvious. For condition 2, we have $\overline{X} \cap L$ is a smooth curve, and we cna check that we have a bijective morphism:

$$\overline{f}'^{-1}(v) \to \overline{X} \cap L$$

induced by ϵ. Actually, pick a point $v \in \mathbf{P}^{n-1}$, the fiber of f' over v is

$$\overline{f}'^{-1}(v) = \left\{ p' \in \overline{X}' \mid f'(p') = v \right\}.$$

Since $f' = \pi|_{\bar{X}'}$ and π came from the linear forms $y_0, \ldots, y_{n-1}$, saying $\pi(p') = v = [v_0 : \cdots : v_{n-1}]$ means precisely

$$\left[y_0 (\epsilon (p')) : \cdots : y_{n-1} (\epsilon (p')) \right] = [v_0 : \cdots : v_{n-1}].$$

If $v = [1 : 0 : \cdots : 0]$, then the condition

$$\left[y_0 (\epsilon (p')) : y_1 (\epsilon (p')) : \cdots : y_{n-1} (\epsilon (p')) \right] = [1 : 0 : \cdots : 0]$$

forces

$$y_1 (\epsilon (p')) = \cdots = y_{n-1} (\epsilon (p')) = 0 \quad \text{and} \quad y_0 (\epsilon (p')) \neq 0.$$

Hence $\epsilon (p')$ indeed lies in

$$\underbrace{\left\{ y_1 = 0, \ldots, y_{n-1} = 0 \right\}}_{=L} \subset \mathbf{P}^N,$$

so $\epsilon (p') \in L = H_1 \cap \cdots \cap H_{n-1}$. Moreover, since $p' \in \overline{X}'$, we also have $\epsilon (p') \in \overline{X}$. Altogether, $\epsilon (p') \in \overline{X} \cap L$.

$$\overline{f}'^{-1}(v) \longrightarrow \overline{X} \cap L, \quad p' \mapsto \epsilon (p').$$

If $v = [v_0 : \cdots : v_{n-1}]$ is more general, one often makes a projective change of coordinates in $\mathbf{P}^{n-1}$ so that v becomes $[1 : 0 : \cdots : 0]$. The map is bijective since $\overline{X} \cap L \cap C = \varnothing$ and away from the blow-up center C the map ϵ is an isomorphism, so we know $\overline{f'}^{-1}(v) \longrightarrow \overline{X} \cap L$ is bijective.

To verify that $\overline{f'}$ is smooth above a neighborhood of v, it suffices, by Hironaka's lemma [SGA1, III. 2.6 (ii)], to check that it is smooth at the generic point of $f'^{-1}(v)$. At this point, $\overline{X}'$ is isomorphic to $\overline{X}$, and the morphism is smooth because L intersects $\overline{X}$ transversely.

It remains to show that g' is étale in a neighborhood of v.

Since Y has dimension $n-1$, the fiber of $g')$ is generically of dimension 0. But as g' is proper hence closed, each fiber of g' is non-empty. We have

$$Y' = \epsilon^{-1}(Y) \amalg D_1 \amalg \cdots \amalg D_r,$$

where $\overline{X} \cap C = \{P_1, \ldots, P_r\}$, and D_i is the blow-up of P_i in $\overline{X}$. Each D_i is identified with $\epsilon^{-1}(P_i)$ and its fiber is mapped isomorphically onto $\mathbf{P}^{n-1}$. It is defined at each point of Y, and the induced morphism on Y is nothing but $g'_{\epsilon^{-1}(Y)}$. The map is étale above v because L intersects Y transversely, and thus g' is étale above v.

Thus, we conclude that there exists a good neighborhood of x.

□

References

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